Chapter 3

Average velocity, Average Rates of Change, and Secant Lines

In this chapter, we extend the idea of slope of a straight line to a related concept for a curve. We will first encounter the idea of an average slope, (also denoted average rate of change, and average velocity in the example to come). That quantity is the slope of a secant line (a line that connects two points on a given curve). We will then consider how a process of refinement can lead us to the concept of the derivative, which is the slope of a tangent line (also called instantaneous rate of change, or instantaneous velocity in the following example). The connection between these two concepts will form a major theme in this course[1]. We begin with a motivating example in which such ideas can be developed naturally, the motion of a falling object.

3.1 Observations about falling objects

The left panel of Figure 3.1 shows a set of three stroboscopic images combined (for visualization purposes) on a single graph. Each set of dots shows successive vertical positions of an object falling from a height of 20 meters over a 2 second time period. In the first image, at left, the location of the ball is given at times $t = 0, 0.5, 1, 1.5$, and 2.0 seconds, i.e. at intervals of $\Delta t = 0.5$ seconds. (A strobe flashing five times, once every $\Delta t = 0.5$ would produce this data.)

We might wonder where the ball is located at times between these successive measurements. Did it vanish? Did it continue in a straight or a looped path? To find out what happened during the intervals between data points, we could increase the strobe frequency, and record measurements more often: for example, in the image at the center of the right panel in Figure 3.1 measurements were made for $t = 0, 0.2, 0.4, \ldots, 2.0$ seconds, i.e. at intervals $\Delta t = 0.2$ seconds. An even closer set of points appears at right, where the time interval between strobe flashes was decreased to $\Delta t = 0.1$ second. By determining the position of the ball at closer time points, we can determine the trajectory of the ball with greater accuracy. The idea of making measurements at finer and finer time increments is important in this example. We will return to it often in our goal of understanding rates of change of natural processes.

In this chapter, we would like to establish some understanding of the idea of a velocity. **Uniform motion** is defined as motion in which a constant distance is covered in constant time intervals. For particles moving uniformly, velocity is constant, and is simply the distance travelled divided by

[1]This connection will form the theme of Lab 3 in this Calculus course

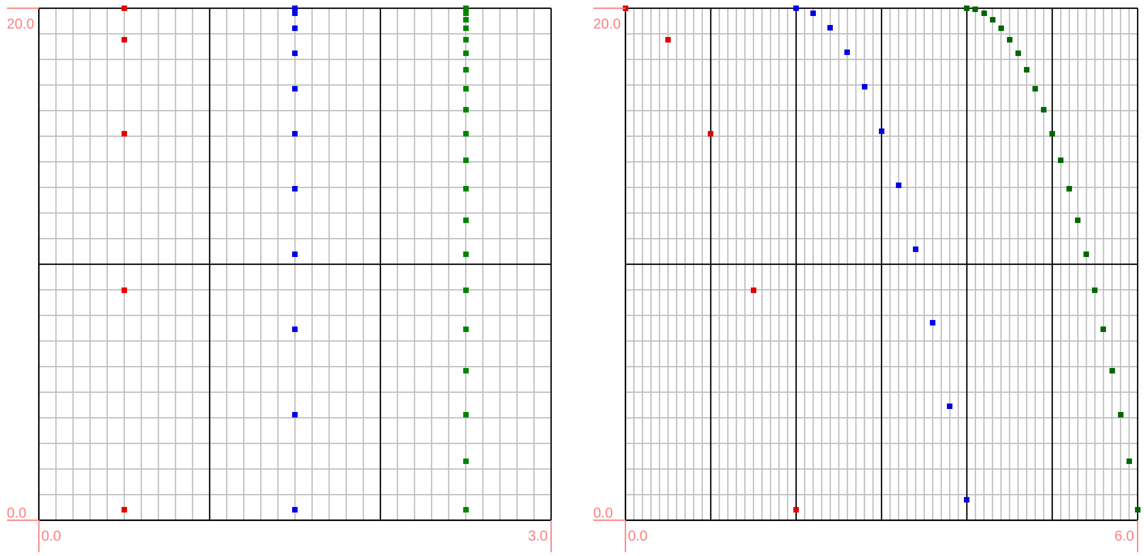

Figure 3.1: Left panel: Three experiments with a falling object are shown here. The time interval between successive positions of the object is $\Delta t = 0.5$ in the experiment shown on the left, $\Delta t = 0.2$ in the middle set, and $\Delta t = 0.1$ in the set at the right. Right panel: The positions of the ball are plotted versus time in each of three experiments. We have decreased the time between strobe flashes: $\Delta t = 0.5, 0.2, 0.1$ for the trials (from left to right).

the time taken. (In uniform motion, velocity does not change over time.) Most types of motion that occur in natural systems is not so simple: as our last example in Figure 3.1 has shown, falling objects cover *increasing* distances as time progresses: they accelerate, meaning that their velocity changes with time. In this situation, we have to rethink how to define the notion of velocity at a given time, and we have to formulate more precisely how we will calculate it. Such questions lead us to the central idea in this chapter: the definition of *average* and *instantaneous* velocity.

Before we define velocity, let us consider the data for our falling ball in a slightly more convenient form, that of a graph. In the right panel of Figure 3.1 Here we have added a time axis for each of the sequences, so that the position(on the vertical axis) and time horizontal axes) are plotted together. Again for ease of visualization, we have combined three possible experiments on the same grid.

3.2 Velocity

We will use the following notation:

t = time,

H = initial height of the ball,

$D(t)$ = height of the ball at time t,

Δt = an interval of time,

ΔD = a change in the vertical position of the ball. Comment: A change in height or in position is called the **displacement**.

Naturally, there is another way of looking at the same data, using a slightly different coordinate

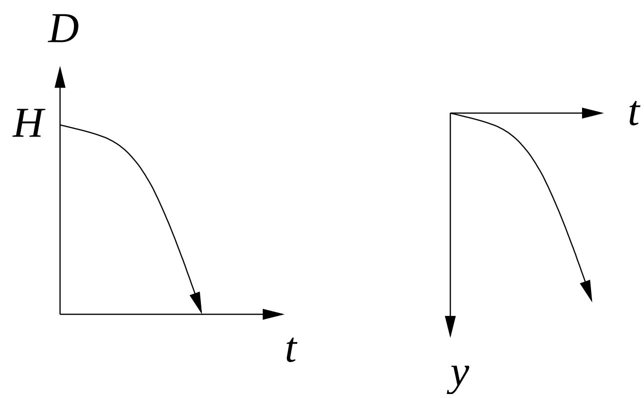

Figure 3.2: The relationship between the two coordinate systems for the falling ball.

system, in which the initial position of the ball is take to be the origin, and the axis points in the direction of the motion. This would lead to the slightly altered definitions

$y(t)$ = vertical distance fallen up to time t,

Δy = a change in the distance fallen by the ball.

The two approaches are related, and we compare them below. (We are just looking at the same data in related but distinct ways: the y axis would point in a direction opposite to a D axis.) Let us agree to call $t = 0$ the time at which the ball was dropped. Then $D(0) = H$ and $y(0) = 0$, i.e. the ball is at height H at time $t = 0$, and has not yet moved vertically. The right panel of Figure 3.1 shows points on the graph of D plotted against t, i.e. the function $D(t)$. If we want to get a more complete graph, we could keep refining the measurements to get points that are closer and closer together. Sometimes it is convenient to consider the actual distance moved y (rather than height D) at time t. We note that

$$y(t) = H - D(t).$$

Naturally, a change of height corresponds to a change in the distance fallen, but some caution with signs is needed. (If $y(a) = H - D(a)$, and $y(b) = H - D(b)$, then $y(b) - y(a) = -(D(b) - D(a))$.) With this preparation, we are ready to define the notion of **average velocity**. To use common convention, we will define this concept in terms of the distance variable y.

3.2.1 Average rate of change and average velocity

Given a time interval, $a \le t \le b$ we will define the *average rate of change* of D over that time interval as follows:

$$\bar{r} = \frac{\text{Change in D}}{\text{Time taken}} = \frac{\Delta D}{\Delta t} = \frac{D(b) - D(a)}{b - a}.$$

For example, if we consider some time t_0 and a later time $t_0 + h$ then the average rate of change between these times is:

$$\bar{r} = \frac{\Delta D}{\Delta t} = \frac{D(t_0 + h) - D(t_0)}{t_0 + h - t_0} = \frac{D(t_0 + h) - D(t_0)}{h}.$$

We can put a geometric meaning to this formula by making the following observation. On our graphs of the position of a falling object versus time, let us connect successive points by straight lines, as shown in Figure 3.3. Then we say that:

The average rate of change of the height D between any two time points is just the slope of the straight line connecting the corresponding points on the graph of $D(t)$.

Because $y(t)$ and $D(t)$ are related, we will shortly see that the average velocity $\bar{v}$ is itself a slope. It has the same magnitude as the slopes of the line segments in Figure 3.3, but the sign is opposite.

We define the **Average velocity**, $\bar{v}$, as follows:

$$\bar{v} = \frac{\text{Distance travelled}}{\text{Time taken}} = \frac{\Delta y}{\Delta t} = \frac{y(b) - y(a)}{b - a}.$$

For example, if we consider some time t_0 and a later time $t_0 + h$ then the average velocity between these times is:

$$\bar{v} = \frac{\Delta y}{\Delta t} = \frac{y(t_0 + h) - y(t_0)}{t_0 + h - t_0} = \frac{y(t_0 + h) - y(t_0)}{h}.$$

From previous remarks, about the relationship of y and D, it follows that $\bar{v} = -\bar{r}$.

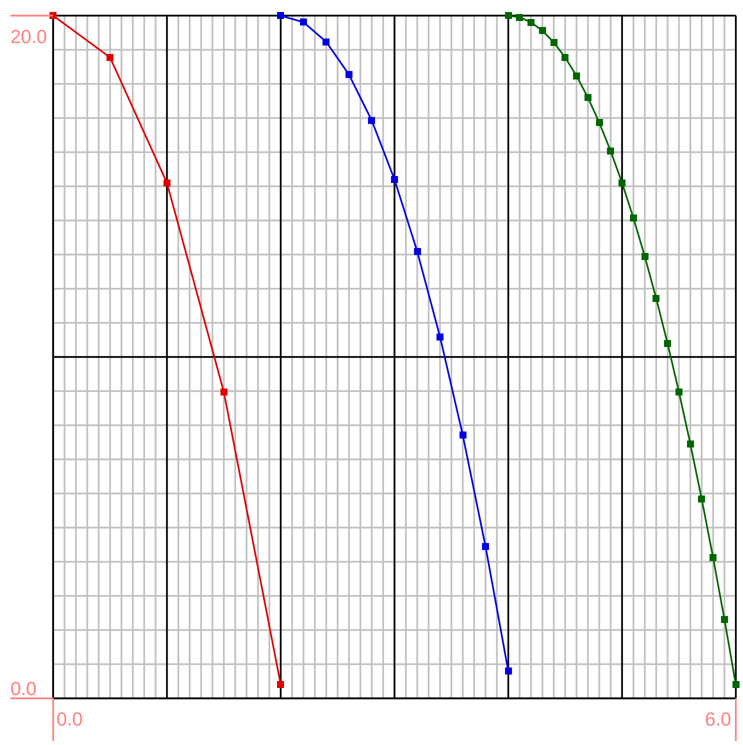

Figure 3.3: The straight lines connecting the points on our graph are called secant lines.

Observations recording the position of a falling object were made long ago by Galileo. He devised some ingenious experiments in which he was able to uncover an interesting relationship between the total distance that an object (falling under the force of gravity) moves during a given total time. A description of Galileo's work and his results appears separately (in lecture notes by Casselman).

Galileo realized that a simple relationship exists between the distance fallen by an object under the effect of gravity and the time taken. Galileo discovered that the distance fallen under the effect of gravity was proportional to the square of the time, i.e., that

$$y(t) = ct^2,$$

where c is a constant. We recognize this quadratic relationship as a simple power function with a constant coefficient. (Later in this course, we will see that this follows directly from the fact that gravity causes constant acceleration - but Galileo, did not realize this fact, nor did he have a clear idea about what acceleration meant.)

When precise measurements are carried out, with units of meters (m) for the distance, and seconds (s) for the time, then it is found that $c = 4.9\text{m/s}^2$. Although Galileo did not have formulae or graph-paper in his day, (and was thus forced to express this relationship in a cumbersome verbal way), what he had discovered was quite remarkable. Using this relationship, we could determine the position of the ball (at least approximately, since nature is hardly ever exact) at any instant.

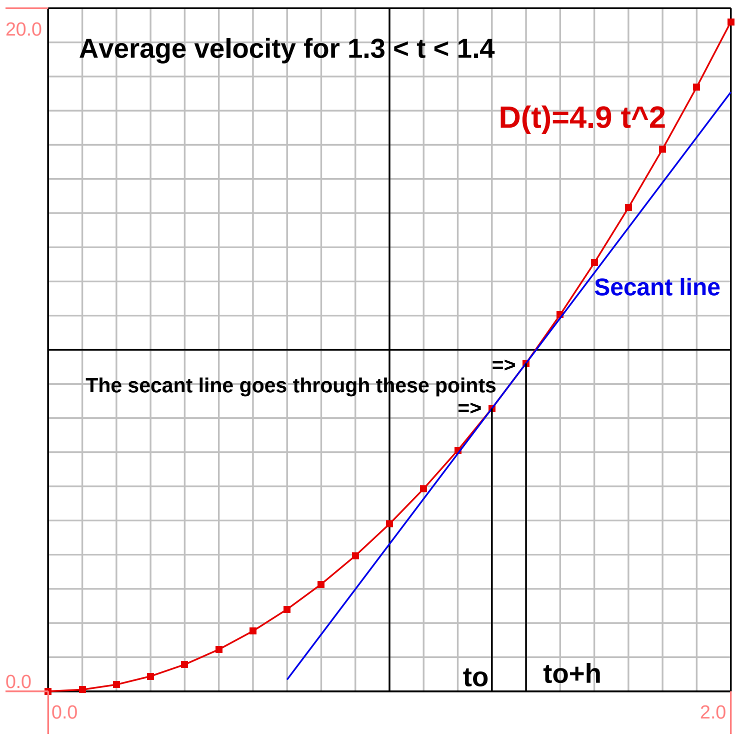

Figure 3.4: The average velocity between time $t_0 + h$ and t_0 is the slope of the secant line shown on this graph.

In Figure 3.4, we have sketched the relationship

$$y = 4.9t^2.$$

On our sketch, we have superimposed a straight line connecting two points on the graph, one at time t_0 and another a little later, at time $t_0 + h$. We call this line a **secant line**. From our definition of $\bar{v}$, it is clear that the slope of the secant line corresponds to the same idea as the average velocity. This concept is also identical to the idea of the **average rate of change** of the position with respect to time over the given time interval.

Some of the points shown in Figure 3.4 are also indicated in the table. We can use these values to determine both the slope of the secant line shown in the figure, and the equation of the secant line.

t	$y(t) = 4.9t^2$
0.0000	0.0000
0.1000	0.0490
0.2000	0.1960
⋮	⋮
1.0000	4.9000
1.1000	5.9290
1.2000	7.0560
1.3000	8.2810
1.4000	9.6040
1.5000	11.0250
⋮	⋮
1.9000	17.6890
2.0000	19.6000

Calculating the average velocity:
The two points on Figure 3.4 through which the secant line is drawn are $(1.3, 8.2810)$ and $(1.4, 9.6040)$. The slope of the line is thus

$$m_{\text{secant}} = \frac{\Delta y}{\Delta t} = \frac{9.6040 - 8.2810}{1.4 - 1.3} = 13.23$$

Thus the average velocity for $1.3 \le t \le 1.4$ seconds is

$$\bar{v} = 13.23 \text{ m/s}.$$

To find the equation of the secant line (shown in Figure 3.4), we note that $(y - 9.604)/(x - 1.4) = 13.23$, or simply

$$y = 13.23x - 2.226.$$

We might want to compute an average velocity between some other pair of points on the graph. Let us examine the same calculation in a more general setting:

Example:

Consider a falling object. Suppose that the total distance fallen at time t is given by

$$y(t) = ct^2.$$

Find the average velocity $\bar{v}$, of the object over the time interval $t_0 \le t \le t_0 + h$.

Solution:

$$\begin{aligned}\bar{v} &= \frac{y(t_0+h)-y(t_0)}{h}\\ &= \frac{c(t_0+h)^2-c(t_0)^2}{h}\\ &= c\left(\frac{(t_0^2+2ht_0+h^2)-(t_0^2)}{h}\right)\\ &= c\left(\frac{2ht_0+h^2}{h}\right)\\ &= c(2t_0+h)\end{aligned} \tag{3.1}$$

Thus the average velocity over the time interval $t_0 < t < xt_0 + h$ is $\bar{v} = c(2t_0 + h)$.

3.2.2 Instantaneous velocity

To arrive at a notion of an instantaneous velocity at some time t_0, we will consider defining average velocities over time intervals $t_0 \le t \le t_0 + h$, that get smaller and smaller: For example, we might make the strobe flash faster, so that the time between flashes, $\Delta t = h$ decreases. (We use the notation $h \to 0$ to denote the fact that we are interested in shrinking the time interval.)

At each stage, we can calculate an average velocity, $\bar{v}$ as described above. As the interval between measurements gets smaller, i.e the process of refining our measurements continues, we arrive at a number that we will call *the instantaneous velocity*. This number represents "the velocity of the ball at the very instant $t = t_0$".

More precisely,

$$\text{Instantaneous velocity} = v(t_0) = \lim_{h\to 0} \frac{y(t_0+h)-y(t_0)}{h}.$$

Example:

Find the instantaneous velocity the same falling object at time t_0.

Solution:
According to our definition, we must determine

$$v(t_0) = \lim_{h\to 0} \frac{y(t_0+h) - y(t_0)}{h}$$

Our calculation would be nearly identical, but for a final step of taking the limit as h, the interval between time-points shrinks to zero:

$$\begin{aligned} v(t_0) &= \lim_{h\to 0} \frac{y(t_0+h) - y(t_0)}{h} \\ &= \lim_{h\to 0} \frac{c(t_0+h)^2 - c(t_0)^2}{h} \\ &= \lim_{h\to 0}\ c\left(\frac{(t_0^2 + 2ht_0 + h^2) - (t_0^2)}{h}\right) \\ &= \lim_{h\to 0}\ c\left(\frac{2ht_0 + h^2}{h}\right) \\ &= \lim_{h\to 0}\ c(2t_0 + h) \\ &= c(2t_0) = 2ct_0 \end{aligned} \tag{3.2}$$

Remarks: In our final steps, we have allowed h to shrink. In the limit, and $h \to 0$, we obtain the instantaneous velocity, i.e. $v(t_0) = 2ct_0$.

3.3 Average and instantaneous rates of change

The development of the ideas that lead to the concept of velocity for a moving object can be generalized to rates of change in many other settings. Suppose $y = f(x)$ is a given function. We define:

The average rate of change is the slope of secant line

The average rate of change of a function between two points on its graph (i.e., the **slope of a secant line**), is

$$\frac{\Delta y}{\Delta x}$$

where Δy is the change in the y value and Δx is the change in the x value.

For example, for the two points x_0 and $x_0 + h$ shown in Figure 3.5(a), we would find

$$\frac{\Delta y}{\Delta x} = \frac{[f(x_0+h) - f(x_0)]}{(x_0+h) - x_0} = \frac{[f(x_0+h) - f(x_0)]}{h}$$

This corresponds to the slope of the line shown in Figure 3.5(b)

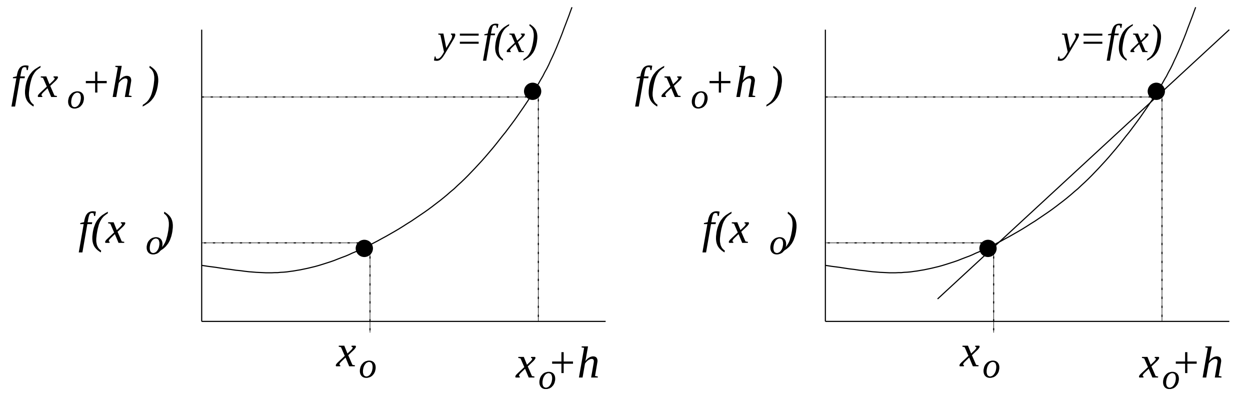

Figure 3.5:

The derivative

The derivative of a function at a point x_0 is denoted $f'(x_0)$ and defined as

$$f'(x_0) = \lim_{h \to 0} \frac{[f(x_0 + h) - f(x_0)]}{h}$$

We recognize from this definition that the derivative is the "instantaneous rate of change" of the function with respect to the variable x at the point of interest, x_0. Another notation used for the derivative is

$$\left.\frac{dy}{dx}\right|_{x_0}.$$

Sometimes, we use secant lines to approximate tangent lines. Indeed, if the distance $h = \Delta x$ between two points of interest is very small, a secant line through those points will be very similar to the corresponding tangent line to the graph at one or the other of those points. This idea that a secant line can approximate a tangent line will reappear later on in the course, when we use numerical approximation methods[2].

3.4 Tangent lines: zooming in on the graph of a function

Another approach to the idea of the derivative is based on the following geometric idea. Consider the graph of some function, and pick some point on that graph. In the example in Figure 3.6 we have shown a graph of the function $y = f(x) = x^3 - x$ and a point shown by a heavy (red) dot.

Now zoom into the selected point, looking at ever higher magnification. (This is shown in the sequence of zooms in Figure 3.7). Eventually, as we get closer to the point of interest, the hills and valleys on the graph disappear off screen, and we start to feel that we live in a much flatter world. In fact, locally, the graph looks more and more like a straight line.

We will refer to this straight line as the **tangent line** to the graph of the function at the given point. The slope of this tangent line will be what we refer to as the **derivative** of the function, at the given point. In the example of Figure 3.7, the slope of the line shown at the bottom panel is roughly 4.

[2]This idea is also explored by the student in Lab 3 of this calculus course

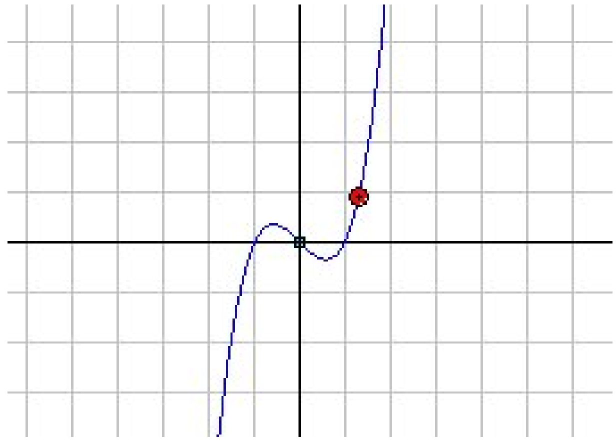

Figure 3.6: A point on the graph of the function $y = f(x) = x^3 - x$.

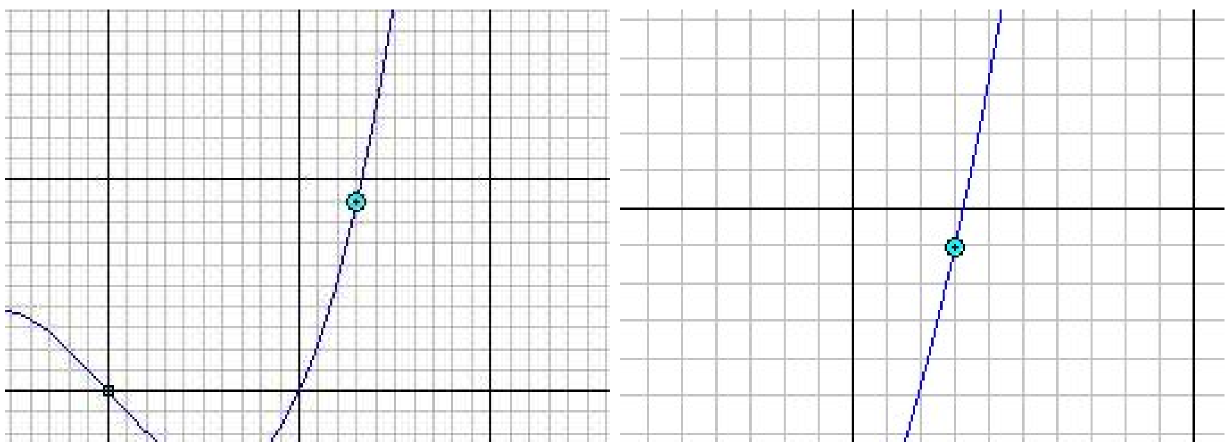

Figure 3.7: Zooming in on a point on the graph of the function $y = f(x) = x^3 - x$.

Clearly the configuration of the tangent line will depend on the point we chose to zoom into. Its slope will also vary from place to place. **For this reason, the derivative, denoted $f'(x)$ is, itself, a function**[3].

In Figure 3.8 we show a zoom into the origin on the graph of the function

$$y = \sin(x).$$

We see from this graph that the slope of the line that we obtain as we zoom into $x = 0$ is $m = 1$. We say that the derivative of the function $y = f(x) = \sin(x)$ at $x = 0$ is 1. We also observe that the line shown in the final image in the sequence of Figure 3.8 is the tangent line to the curve at $x = 0$. The equation of this line is simply $y = x$. (This follows from the fact that the line has slope 1 and goes through the point $(0, 0)$.) We can also say that close to $x = 0$ the graph of $y = \sin(x)$ looks a lot like the line $y = x$. This is equivalent to saying that

$$\sin(x) \approx x, \quad \text{or} \quad \frac{\sin(x)}{x} \approx 1$$

[3]This is a very important concept, that will be explored throughout the course. This idea appears in Lab 3.

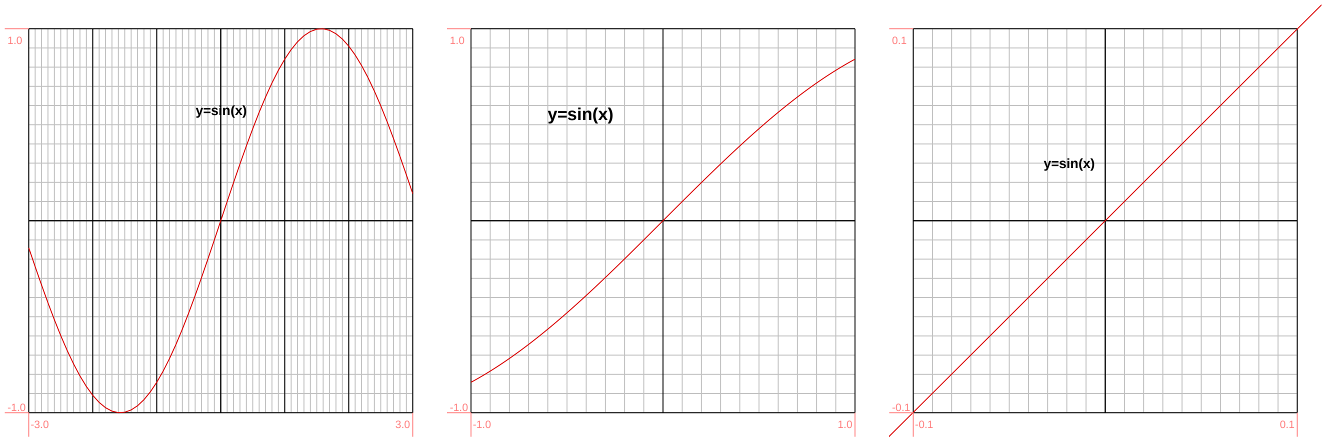

Figure 3.8: Zooming in on the point $x = 0$ on the graph of the function $y = f(x) = \sin(x)$.

for small x, or, more formally, that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

We will find this limit useful in later calculations.

3.5 Sketching the graph of the derivative

In the previous section, we observed that the derivative of a function is itself a function. This prompts us to explore how the two functions are related (both analytically, i.e. using pen and paper computations, and geometrically, by drawing sketches). Here we make the first such geometric connection. This sketching skill will be called upon again and again in the course. We will also find that numerical methods to compute derivatives will be useful[4].

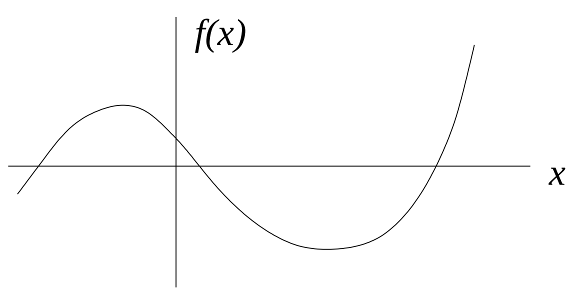

Figure 3.9:

In Figure 3.9 we show the graph of some function, $f(x)$. We would like to sketch the derivative, $f'(x)$ corresponding to this function. (Recall that the derivative is also a function.) Keep in mind that this sketch will be approximate, but will contain some important elements.

[4]Indeed, a spreadsheet tool, used appropriately, can easily produce graphs of a function and its derivatives. This will form one of the concepts explored in parts of Lab 3 in the calculus course.

In Figure 3.10 we start by sketching in a number of tangent lines on the graph of $f(x)$. We will pay special attention to the slopes (rather than height, length, or any other property) of these dashes. Copying these lines in a row along the direction of the x axis, we estimate their slopes with rather approximate numerical values.

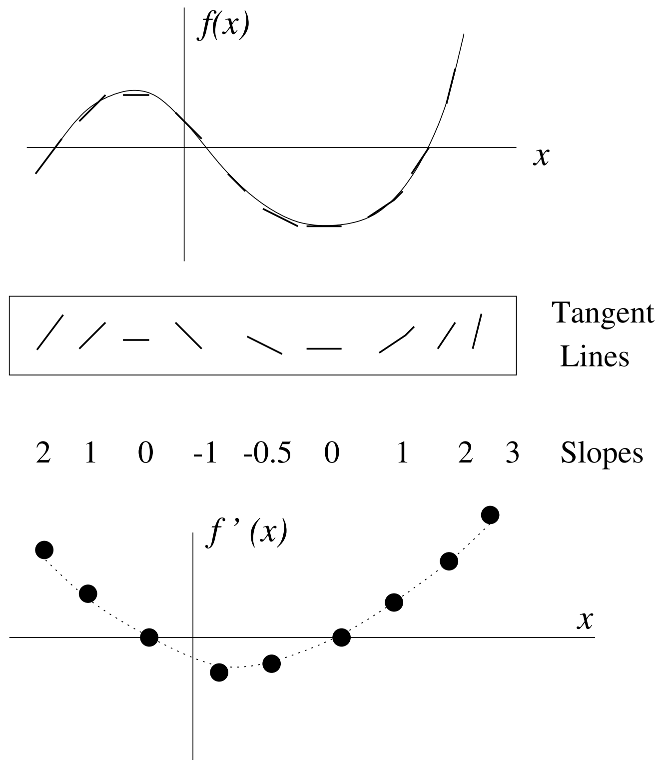

Figure 3.10:

We notice that the slopes start out positive, decrease to zero, become negative, and then increase again through zero back to positive values. (We see precisely two dashes that are horizontal, and so have slope 0.) Next, we plot the numerical values (for slopes) that we have recorded on a new graph. This is the beginning of the graph of the derivative, $f'(x)$. Only a few points have been plotted in our figure of $f'(x)$; we could add other values if we so chose, but the trend, is fairly clear: The derivative function has two zeros (places of intersection with the x axis. It dips down below the axis between these places. In Figure 3.11 we show the original function $f(x)$ and its derivative $f'(x)$ now drawn as a continuous curve. We have aligned these graphs so that the slope of $f(x)$ matches the value of $f'(x)$ shown directly below.

3.6 Some technical matters: limits

We have surreptitiously introduced some notation involving limits without carefully defining what was meant. In this section, such technical matters are briefly discussed.

The concept of a **limit** helps us to describe the behaviour of a function close to some point of interest. This proves to be most useful in the case of functions that are either not continuous, or

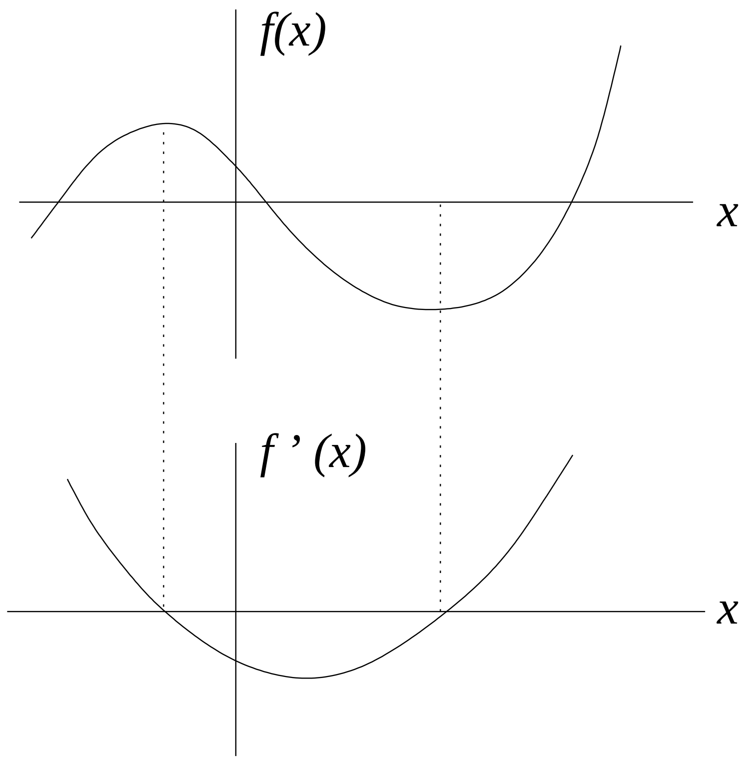

Figure 3.11:

not defined somewhere. We will use the notation

$$\lim_{x \to a} f(x)$$

to denote the value that the function f approaches as x gets closer and closer to the value a. If $x = a$ is a point at which the function is defined and continuous (informally, has no "breaks in its graph") the value of the limit and the value of the function at a point are the same, i.e.

If f is continuous at $x = a$ then

$$\lim_{x \to a} f(x) = f(a).$$

Many of the functions considered so far in this course have been continuous at all their points. For example, power functions and polynomials are continuous everywhere.

3.6.1 Example:

Find $\lim_{x \to 2} f(x)$ for the function $y = f(x) = 2x^2 - x^3$.

Solution:

Since this function is a polynomial, and so continuous everywhere, we can simply plug in the relevant value of x, i.e.

$$\lim_{x \to 2} \left(2x^2 - x^3\right) = 2 \cdot 2^2 - 2^3 = 0.$$

Thus when x gets closer to 2, the value of the function gets closer to 0. (In fact, the value of the limit is the same as the value of the function at the given point.)

3.6.2 Example:

Find $\lim_{x\to 0} f(x)$ for the function $y = f(x) = 10$

Solution:

This function is constant (and continuous) everywhere. We conclude immediately that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} 10 = 10.$$

3.6.3 Example:

Find $\lim_{x\to 0} f(x)$ for the function $y = f(x) = \sin(x)$.

Solution:

This function is a continuous trigonometric function, and has the value $\sin(0) = 0$ at the origin. Thus

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \sin(x) = 0$$

3.6.4 Example:

Find $\lim_{x\to \pi/2} f(x)$ for the function $y = f(x) = \tan(x)$.

Solution:

The function $\tan(x) = \sin(x)/\cos(x)$ cannot be continuous at $x = \pi/2$ because $\cos(x)$ in the denominator takes on the value of zero at the point $x = \pi/2$. Moreover, the value of this function becomes unbounded (grows without a limit) as $x \to \pi/2$. We say in this case that "the limit does not exist". We sometimes use the notation

$$\lim_{x\to \pi/2} \tan(x) = \pm\infty.$$

(We can distinguish the fact that the function approaches $+\infty$ as x approaches $\pi/2$ from below, and $-\infty$ as x approaches $\pi/2$ from higher values.

In the previous examples, evaluating the limit, where it existed, was as simple as plugging the appropriate value of x into the function itself. The next example shows that this is not always possible, particularly in the case of functions that have a "hole".

3.6.5 Example:

Find $\lim_{x\to 2} f(x)$ for the function $y = f(x) = (x - 2)/(x^2 - 4)$.

Solution:

If $x \neq 2$ then $f(x) = (x-2)/(x^2-4) = (x-2)/(x-2)(x+2)$ takes on the same values as the expression $1/(x+2)$. At the point $x = 2$, the function $f(x)$ is not defined, since we are not allowed division by zero. However, the limit of this function does exist:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x-2)}{(x^2-4)}.$$

Provided $x \neq 2$ we can factor the denominator and cancel:

$$\lim_{x\to 2} \frac{(x-2)}{(x^2-4)} = \lim_{x\to 2} \frac{(x-2)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{(x+2)}$$

Now we can substitute $x = 2$ to obtain

$$\lim_{x\to 2} f(x) = \frac{1}{(2+2)} = \frac{1}{4}$$

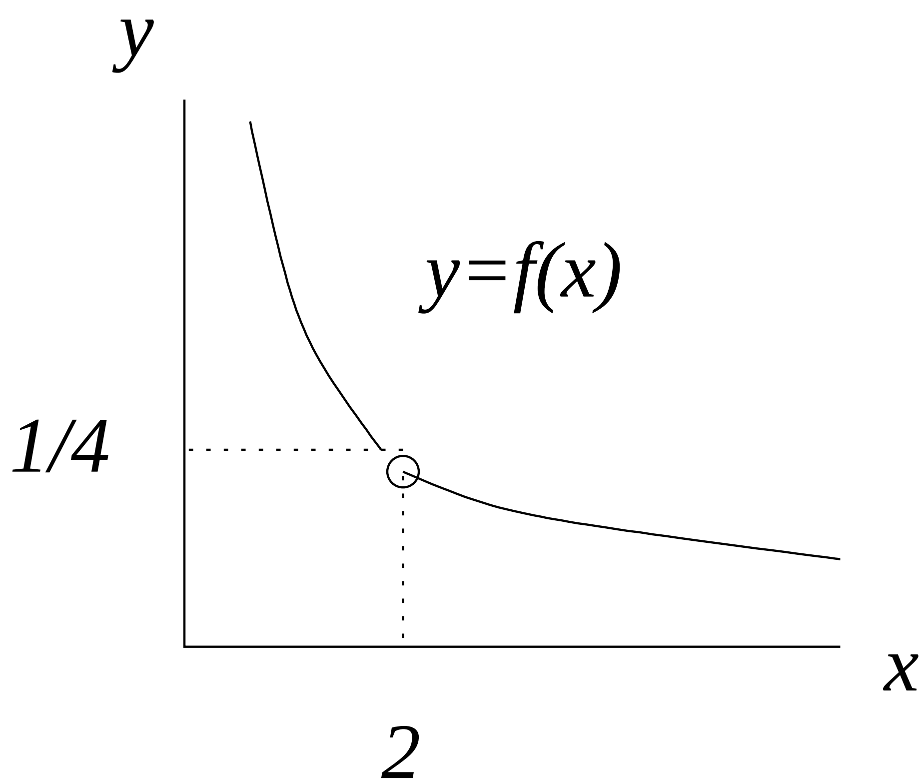

Figure 3.12: The function $y = \frac{(x-2)}{(x^2-4)}$ has a "hole in its graph at $x = 2$. The limit of the function as x approaches 2 does exist, and "supplies the missing point": $\lim_{x\to 2} f(x) = \frac{1}{4}$.

We can also describe the behaviour "at infinity" i.e. the trend displayed by a function for very large (positive or negative) values of x. We consider a few examples of this sort below.

3.6.6 Example:

Find $\lim_{x\to\infty} f(x)$ for the function $y = f(x) = x^3 - x^5 + x$.

Solution:

All polynomials grow in an unbounded way as x tends to very large values. We can determine whether the function approaches positive or negative unbounded values by looking at the coefficient of the highest power of x, since that power dominates at large x values. In this example, we find that the term $-x^5$ is that highest power. Since this has a negative coefficient, the function will approach unbounded negative values as x gets larger in the positive direction, i.e.

$$\lim_{x\to\infty} x^3 - x^5 + x = \lim_{x\to\infty} -x^5 = -\infty.$$

3.7 Secant lines using the spreadsheet

We can use the Spreadsheet to plot the function

$$y = f(x) = x^2 \qquad 0 \leq x \leq 3,$$

and show its secant line through the points $x = 1$ and $x = 2$ on the same graph. It can be shown that the equation of the desired secant line is

$$y = 3x - 2.$$

The function and its secant line are shown on figure 3.13. More details on this calculation and spreadsheet operations that produced this graph are given in Lab 3.

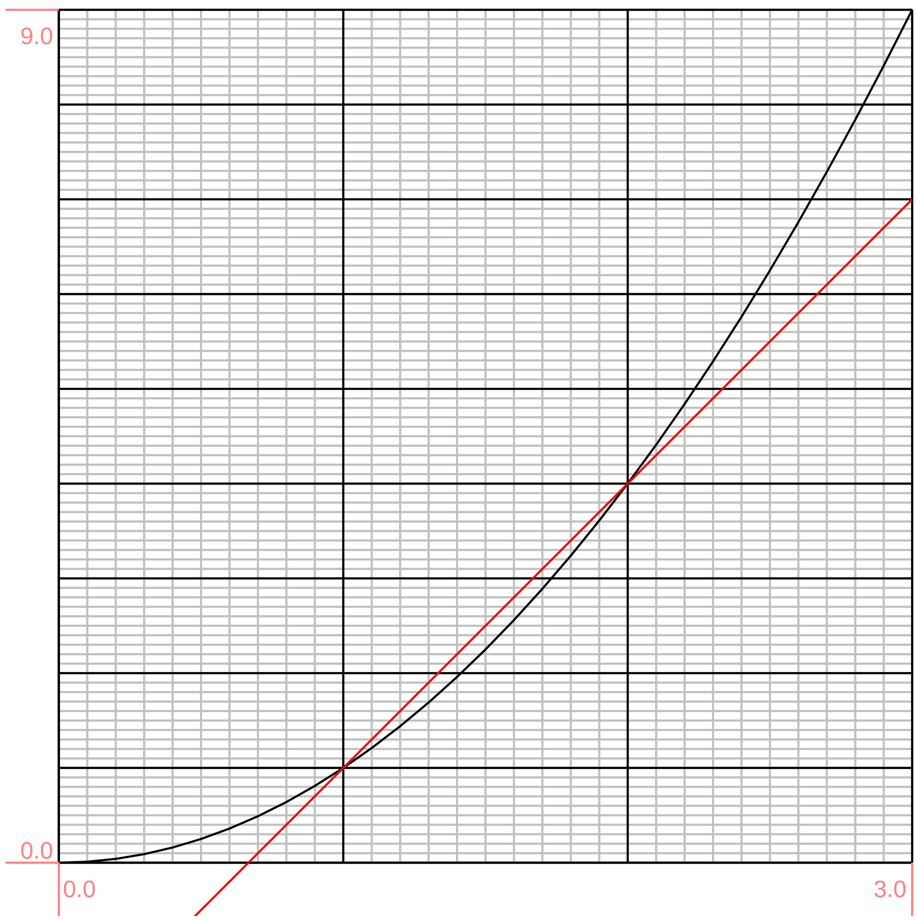

Figure 3.13: The graph of the function $y = f(x) = x^2$, showing one of its secant lines.