Chapter 1

Areas, volumes and simple sums

1.1 Introduction

This introductory chapter has several aims. First, we concentrate here a number of basic formulae for areas and volumes that are used later in developing the notions of integral calculus. Among these are areas of simple geometric shapes and formulae for sums of certain common sequences. An important idea is introduced, namely that we can use the sum of areas of elementary shapes to approximate the areas of more complicated objects, and that the approximation can be made more accurate by a process of refinement.

We show using examples how such ideas can be used in calculating the volumes or areas of more complex objects. In particular, we conclude with a detailed exploration of the structure of branched airways in the lung as an application of ideas in this chapter.

1.2 Areas of simple shapes

One of the main goals in this course will be calculating areas enclosed by curves in the plane and volumes of three dimensional shapes. We will find that the tools of calculus will provide important and powerful techniques for meeting this goal. Some shapes are simple enough that no elaborate techniques are needed to compute their areas (or volumes). We briefly survey some of these simple geometric shapes and list what we know or can easily determine about their area or volume.

The areas of simple geometrical objects, such as rectangles, parallelograms, triangles, and circles are given by elementary formulae. Indeed, our ability to compute areas and volumes of more elaborate geometrical objects will rest on some of these simple formulae, summarized below.

Rectangular areas

Most integration techniques discussed in this course are based on the idea of carving up irregular shapes into rectangular strips. Thus, areas of rectangles will play an important part in those methods.
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- The area of a rectangle with base \( b \) and height \( h \) is
  \[ A = b \cdot h \]

- Any parallelogram with height \( h \) and base \( b \) also has area, \( A = b \cdot h \). See Figure 1.1(a) and (b)

\[ \text{(a)} \quad \text{(b)} \]

\[ \begin{array}{c}
\text{(c)} \\
\text{(d)} \\
\text{(e)} \\
\text{(f)}
\end{array} \]

Figure 1.1. Planar regions whose areas are given by elementary formulae.

Areas of triangular shapes

A few illustrative examples in this chapter will be based on dissecting shapes (such as regular polygons) into triangles. The areas of triangles are easy to compute, and we summarize this review material below. However, triangles will play a less important role in subsequent integration methods.

- The area of a triangle can be obtained by slicing a rectangle or parallelogram in half, as shown in Figure 1.1(c) and (d). Thus, any triangle with base \( b \) and height \( h \) has area
  \[ A = \frac{1}{2}bh. \]
1.2. Areas of simple shapes

• In some cases, the height of a triangle is not given, but can be determined from other information provided. For example, if the triangle has sides of length $b$ and $r$ with enclosed angle $\theta$, as shown on Figure 1.1(e) then its height is simply $h = r \sin(\theta)$, and its area is

$$A = \frac{1}{2}br \sin(\theta)$$

• If the triangle is isosceles, with two sides of equal length, $r$, and base of length $b$, as in Figure 1.1(f) then its height can be obtained from Pythagoras’s theorem, i.e. $h^2 = r^2 - (b/2)^2$ so that the area of the triangle is

$$A = \frac{1}{2}b\sqrt{r^2 - (b/2)^2}.$$ 

1.2.1 Example 1: Finding the area of a polygon using triangles: a “dissection” method

Using the simple ideas reviewed so far, we can determine the areas of more complex geometric shapes. For example, let us compute the area of a regular polygon with $n$ equal sides, where the length of each side is $b = 1$. This example illustrates how a complex shape (the polygon) can be dissected into simpler shapes, namely triangles.

![Diagram of an equilateral polygon dissected into triangles](image)

**Figure 1.2.** An equilateral $n$-sided polygon with sides of unit length can be dissected into $n$ triangles. One of these triangles is shown at right. Since it can be further divided into two Pythagorean triangles, trigonometric relations can be used to find the height $h$ in terms of the length of the base $1/2$ and the angle $\theta/2$.

**Solution**

The polygon has $n$ sides, each of length $b = 1$. We dissect the polygon into $n$ isosceles triangles, as shown in Figure 1.2. We do not know the heights of these triangles, but the angle $\theta$ can be found. It is

$$\theta = \frac{2\pi}{n}$$

since together, $n$ of these identical angles make up a total of $360^\circ$ or $2\pi$ radians.

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1This calculation will be used again to find the area of a circle in Section 1.2.2. However, note that in later chapters, our dissections of planar areas will focus mainly on rectangular pieces.
Let $h$ stand for the height of one of the triangles in the dissected polygon. Then trigonometric relations relate the height to the base length as follows:

$$\frac{\text{opp}}{\text{adj}} = \frac{b/2}{h} = \tan(\theta/2)$$

Using the fact that $\theta = 2\pi/n$, and rearranging the above expression, we get

$$h = \frac{b}{2 \tan(\pi/n)}$$

Thus, the area of each of the $n$ triangles is

$$A = \frac{1}{2} bh = \frac{1}{2} b \left( \frac{b}{2 \tan(\pi/n)} \right).$$

The statement of the problem specifies that $b = 1$, so

$$A = \frac{1}{2} \left( \frac{1}{2 \tan(\pi/n)} \right).$$

The area of the entire polygon is then $n$ times this, namely

$$A_{n\text{-gon}} = \frac{n}{4 \tan(\pi/n)}.$$

For example, the area of a square (a polygon with 4 equal sides, $n = 4$) is

$$A_{\text{square}} = \frac{4}{4 \tan(\pi/4)} = \frac{1}{\tan(\pi/4)} = 1,$$

where we have used the fact that $\tan(\pi/4) = 1$.

As a second example, the area of a hexagon (6 sided polygon, i.e. $n = 6$) is

$$A_{\text{hexagon}} = \frac{6}{4 \tan(\pi/6)} = \frac{3}{2(1/\sqrt{3})} = \frac{3\sqrt{3}}{2}.$$

Here we used the fact that $\tan(\pi/6) = 1/\sqrt{3}$.

### 1.2.2 Example 2: How Archimedes discovered the area of a circle: dissect and “take a limit”

As we learn early in school the formula for the area of a circle of radius $r$, $A = \pi r^2$. But how did this convenient formula come about? and how could we relate it to what we know about simpler shapes whose areas we have discussed so far. Here we discuss how this formula for the area of a circle was determined long ago by Archimedes using a clever “dissection” and approximation trick. We have already seen part of this idea in dissecting a polygon into triangles, in Section 1.2.1. Here we see a terrifically important second step that formed the “leap of faith” on which most of calculus is based, namely taking a limit as the number of subdivisions increases.

First, we recall the definition of the constant $\pi$:

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2This idea has important parallels with our later development of integration. Here it involves adding up the areas of triangles, and then taking a limit as the number of triangles gets larger. Later on, we do much the same, but using rectangles in the dissections.
Definition of $\pi$

In any circle, $\pi$ is the ratio of the circumference to the diameter of the circle. (Comment: expressed in terms of the radius, this assertion states the obvious fact that the ratio of $2\pi r$ to $2r$ is $\pi$.)

Shown in Figure 1.3 is a sequence of regular polygons inscribed in the circle. As the number of sides of the polygon increases, its area gradually becomes a better and better approximation of the area inside the circle. Similar observations are central to integral calculus, and we will encounter this idea often. We can compute the area of any one of these polygons by dissecting into triangles. All triangles will be isosceles, since two sides are radii of the circle, whose length we'll call $r$.

![Figure 1.3. Archimedes approximated the area of a circle by dissecting it into triangles.](image)

Let $r$ denote the radius of the circle. Suppose that at one stage we have an $n$ sided polygon. (If we knew the side length of that polygon, then we already have a formula for its area. However, this side length is not known to us. Rather, we know that the polygon should fit exactly inside a circle of radius $r$.) This polygon is made up of $n$ triangles, each one an isosceles triangle with two equal sides of length $r$ and base of undetermined length that we will denote by $b$. (See Figure 1.3.) The area of this triangle is

$$A_{\text{triangle}} = \frac{1}{2}bh.$$ 

The area of the whole polygon, $A_n$, is then

$$A = n \cdot \text{(area of triangle)} = n \frac{1}{2}bh = \frac{1}{2}(nb)h.$$ 

We have grouped terms so that $(nb)$ can be recognized as the perimeter of the polygon (i.e. the sum of the $n$ equal sides of length $b$ each). Now consider what happens when we increase the number of sides of the polygon, taking larger and larger $n$. Then the height of each triangle will get closer to the radius of the circle, and the perimeter of the polygon will get closer and closer to the perimeter of the circle, which is (by definition) $2\pi r$. i.e. as $n \to \infty$,

$$h \to r, \quad (nb) \to 2\pi r$$

so

$$A = \frac{1}{2}(nb)h \to \frac{1}{2}(2\pi r)r = \pi r^2$$
We have used the notation “$\rightarrow$” to mean that in the limit, as $n$ gets large, the quantity of interest “approaches” the value shown. This argument proves that the area of a circle must be

$$A = \pi r^2.$$  

One of the most important ideas contained in this little argument is that by approximating a shape by a larger and larger number of simple pieces (in this case, a large number of triangles), we get a better and better approximation of its area. This idea will appear again soon, but in most of our standard calculus computations, we will use a collection of rectangles, rather than triangles, to approximate areas of interesting regions in the plane.

### Areas of other shapes

We concentrate here the area of a circle and of other shapes.

- The area of a circle of radius $r$ is
  $$A = \pi r^2.$$  

- The surface area of a sphere of radius $r$ is
  $$S_{\text{ball}} = 4\pi r^2.$$  

- The surface area of a right circular cylinder of height $h$ and base radius $r$ is
  $$S_{\text{cyl}} = 2\pi rh.$$  

### Units

The units of area can be meters$^2$ ($m^2$), centimeters$^2$ ($cm^2$), square inches, etc.

### 1.3 Simple volumes

Later in this course, we will also be computing the volumes of 3D shapes. As in the case of areas, we collect below some basic formulae for volumes of elementary shapes. These will be useful in our later discussions.

1. The volume of a cube of side length $s$ (Figure 1.4a), is
   $$V = s^3.$$  

2. The volume of a rectangular box of dimensions $h$, $w$, $l$ (Figure 1.4b) is
   $$V = hwl.$$  

3. The volume of a cylinder of base area $A$ and height $h$, as in Figure 1.4(c), is
   $$V = Ah.$$  

   This applies for a cylinder with flat base of any shape, circular or not.
4. In particular, the volume of a cylinder with a circular base of radius \( r \), (e.g. a disk) is
\[
V = h(\pi r^2).
\]

5. The volume of a sphere of radius \( r \) (Figure 1.4d), is
\[
V = \frac{4}{3} \pi r^3.
\]

6. The volume of a spherical shell (hollow sphere with a shell of some small thickness, \( \tau \)) is approximately
\[
V \approx \tau \cdot (\text{surface area of sphere}) = 4\pi \tau r^2.
\]

7. Similarly, a cylindrical shell of radius \( r \), height \( h \) and small thickness, \( \tau \) has volume given approximately by
\[
V \approx \tau \cdot (\text{surface area of cylinder}) = 2\pi \tau rh.
\]

**Units**

The units of volume are meters\(^3\) (m\(^3\)), centimeters\(^3\) (cm\(^3\)), cubic inches, etc.
1.3.1 Example 3: The Tower of Hanoi: a tower of disks

In this example, we consider how elementary shapes discussed above can be used to determine volumes of more complex objects. The Tower of Hanoi is a shape consisting of a number of stacked disks. It is a simple calculation to add up the volumes of these disks, but if the tower is large, and comprised of many disks, we would want some shortcut to avoid long sums.\footnote{Note that the idea of computing a volume of a radially symmetric 3D shape by dissection into disks will form one of the main themes in Chapter 5. Here, the sums of the volumes of disks is exactly the same as the volume of the tower. Later on, the disks will only approximate the true 3D volume, and a limit will be needed to arrive at a “true volume”.}

![Figure 1.5. Computing the volume of a set of disks. (This structure is sometimes called the tower of Hanoi after a mathematical puzzle by the same name.)](image)

(a) Compute the volume of a tower made up of four disks stacked up one on top of the other, as shown in Figure 1.5. Assume that the radii of the disks are 1, 2, 3, 4 units and that each disk has height 1.

(b) Compute the volume of a tower made up of 100 such stacked disks, with radii \( r = 1, 2, \ldots, 99, 100 \).

Solution

(a) The volume of the four-disk tower is calculated as follows:

\[
V = V_1 + V_2 + V_3 + V_4,
\]

where \( V_i \) is the volume of the \( i \)'th disk whose radius is \( r = i \), \( i = 1, 2 \ldots 4 \). The height of each disk is \( h = 1 \), so

\[
V = (\pi 1^2) + (\pi 2^2) + (\pi 3^2) + (\pi 4^2) = \pi (1 + 4 + 9 + 16) = 30\pi.
\]

(b) The idea will be the same, but we have to calculate

\[
V = \pi (1^2 + 2^2 + 3^2 + \cdots + 99^2 + 100^2).
\]

It would be tedious to do this by adding up individual terms, and it is also cumbersome to write down the long list of terms that we will need to add up. This motivates inventing some helpful notation, and finding some clever way of performing such calculations.
1.4 Sigma Notation

Let’s consider the sequence of squared integers

\[(1^2, 2^2, 3^2, 4^2, 5^2, \ldots) = (1, 4, 9, 16, 25, \ldots)\]

and let’s add them up. Okay so the sum of the first square is \(1^2 = 1\), the sum of the first two squares is \(1^2 + 2^2 = 5\), the sum of the first three squares is \(1^2 + 2^2 + 3^2 = 14\), etc. Now suppose we want to sum the first fifteen square integers—how should we write out this sum in our notes? Of course, the sum just looks like

\[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 + 12^2 + 13^2 + 14^2 + 15^2\]

which equals

\[1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 + 121 + 144 + 169 + 196 + 225.\]

But let’s agree that the above expression is an eyesore. Frankly it occupies more space than it deserves, is confusing to look at, and is logically redundant. All we wanted to do was ‘sum the first fifteen squared integers’, not ‘sum all the phone numbers that occur in the first fifteen pages of the phone book’. A sum with such a simple internal structure should have a simple notation. Mathematicians and physicists have such a notation which is convenient, logical, and simplifies many summations—this is the ‘sigma notation’.

The sum of the elements \(a_k + a_{k+1} + \cdots + a_n\) will be written \(\sum_{j=k}^{n} a_j\), i.e.

\[\sum_{j=k}^{n} a_j := a_k + a_{k+1} + \cdots + a_n.\]

The symbol \(\Sigma\) is the Greek letter for ‘S’—we think of ‘S’ as standing for summation. The expression \(\sum_{j=k}^{n} a_j\) represents the sum of the elements \(a_k, a_{k+1}, \ldots, a_n\). The letter ‘\(j\)’ is the index of summation and is a dummy-variable, i.e. you are free to replace it by any letter or symbol you want (like \(k, \ell, m, n, \heartsuit, \star, \ldots\)). Both \(\sum_{\heartsuit=1}^{2013} a_{\heartsuit}\) and \(\sum_{\triangle=1}^{2013} a_{\triangle}\) stand for the same sum: \(a_1 + a_2 + \cdots + a_{2013}\). The notation \(j = k\) that appears underneath \(\Sigma\) indicates where the sum begins (i.e. which term starts off the series), and the superscript \(n\) tells us where it ends. We will be interested in getting used to this notation, as well as in actually computing the value of the desired sum using a variety of shortcuts.

Simple examples

1. The convenience of the sigma notation is that the above sum of the first fifteen squared integers has now the more compact form

\[\sum_{j=1}^{15} j^2.\]

We shall find below a closed-formula for this sum which will be easier to derive using the sigma notation.
2. Suppose we want to form the sum of ten numbers, each equal to 1. We would write this as

\[ S = 1 + 1 + 1 + \ldots + 1 \equiv \sum_{k=1}^{10} 1. \]

The notation \ldots signifies that we have left out some of the terms (out of laziness, or in cases where there are too many to conveniently write down.) We could have just as well written the sum with another symbol (e.g. \( n \)) as the index, i.e. the same operation is implied by

\[ \sum_{n=1}^{10} 1. \]

To compute the value of the sum we use the elementary fact that the sum of ten ones is just 10, so

\[ S = \sum_{k=1}^{10} 1 = 10. \]

3. Sum of squares: Expand and sum the following:

\[ S = \sum_{k=1}^{4} k^2. \]

**Solution:**

\[ S = \sum_{k=1}^{4} k^2 = 1 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30. \]

(We have already seen this sum in part (a) of The Tower of Hanoi.)

4. Common factors: Add up the following list of 100 numbers (only a few of them are shown):

\[ S = 3 + 3 + 3 + 3 + \ldots + 3. \]

**Solution:** There are 100 terms, all equal, so we can take out a common factor

\[ S = 3 + 3 + 3 + 3 + \ldots + 3 = \sum_{k=1}^{100} 3 = 3 \sum_{k=1}^{100} 1 = 3(100) = 300. \]

5. Find the pattern: Write the following terms in summation notation:

\[ S = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}. \]

**Solution:** We recognize that there is a pattern in the sequence of terms, namely, each one is \( \frac{1}{3} \) raised to an increasing integer power, i.e.

\[ S = \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4. \]
We can represent this with the “Sigma” notation as follows:

\[ S = \sum_{n=1}^{4} \left( \frac{1}{3} \right)^n. \]

The “index” \( n \) starts at 1, and counts up through 2, 3, and 4, while each term has the form of \((1/3)^n\). This series is a geometric series, to be explored shortly. In most cases, a standard geometric series starts off with the value 1. We can easily modify our notation to include additional terms, for example:

\[ S = \sum_{n=0}^{5} \left( \frac{1}{3} \right)^n = 1 + \frac{1}{3} + \left( \frac{1}{3} \right)^2 + \left( \frac{1}{3} \right)^3 + \left( \frac{1}{3} \right)^4 + \left( \frac{1}{3} \right)^5. \]

Learning how to compute the sum of such terms will be important to us, and will be described later on in this chapter.

6. Often there are different but equivalent ways to represent the same sum:

\[ 4 + 5 + 6 + 7 + 8 = \sum_{j=4}^{8} j = \sum_{j=1}^{5} (3 + j). \]

7. It is useful to have some dexterity in arranging and rearranging the Sigma notations. For instance, the sum

\[ 1 - 2 + 3 - 4 + 5 - 6 + \cdots \]

has no upper bound and we may write

\[ 1 - 2 + 3 - 4 + 5 - 6 + \cdots = \sum_{j=1}^{\infty} (-1)^{j+1} j = \sum_{j=0}^{\infty} (-1)^j (j + 1) \]

to highlight the fact that this sum has infinitely many terms.

8.\n
\[ 7 + 9 + 11 + 13 + 15 = \sum_{j=0}^{4} (7 + 2j). \]

9. \[ \sum_{j=n}^{n} a_j = a_n. \]

10. \[ \sum_{j=10}^{1} a_j = 0. \]
Manipulations of sums

Since addition is commutative and distributive, sums of lists of numbers satisfy many convenient properties. We give a few examples below:

- Simplify the following expression:
  \[ \sum_{k=1}^{10} 2^k - \sum_{k=3}^{10} 2^k. \]

\textit{Solution:} We have
\[ \sum_{k=1}^{10} 2^k - \sum_{k=3}^{10} 2^k = (2 + 2^2 + 2^3 + \cdots + 2^{10}) - (2^3 + \cdots + 2^{10}) = 2 + 2^2. \]

Alternatively we could have arrived at this conclusion directly from
\[ \sum_{k=1}^{10} 2^k - \sum_{k=3}^{10} 2^k = \sum_{k=1}^{2} 2^k = 2 + 2^2 = 2 + 4 = 6. \]

The idea is that all but the first two terms in the first sum will cancel. The only remaining terms are those corresponding to \( k = 1 \) and \( k = 2 \).

- Expand the following expression:
  \[ \sum_{n=0}^{5} (1 + 3^n). \]

\textit{Solution:} We have
\[ \sum_{n=0}^{5} (1 + 3^n) = \sum_{n=0}^{5} 1 + \sum_{n=0}^{5} 3^n. \]

1.4.1 Formulae for sums of integers, squares, and cubes

The general formulae are:

\[ \sum_{k=1}^{N} k = \frac{N(N+1)}{2}, \quad (1.1) \]
\[ \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6} \quad (1.2) \]
\[ \sum_{k=1}^{N} k^3 = \left( \frac{N(N+1)}{2} \right)^2. \quad (1.3) \]

We now provide a justification as to why these formulae are valid. The sum of the first \( N \) integers can perhaps most easily be seen by the following amusing argument.
1.4. Sigma Notation

The sum of consecutive integers (Gauss’ formula)

We first show that the sum $S_N$ of the first $N$ integers is

$$S_N = 1 + 2 + 3 + \cdots + N = \sum_{k=1}^{N} k = \frac{N(N + 1)}{2}. \quad (1.4)$$

The following trick is due to Gauss. By aligning two copies of the above sum, one written backwards, we can easily add them up one by one vertically. We see that:

$$S_N = 1 + 2 + \cdots + (N - 1) + N$$
$$+ \quad S_N = N + (N - 1) + \cdots + 2 + 1$$

$$2S_N = (1 + N) + (1 + N) + \cdots + (1 + N) + (1 + N)$$

Thus, there are $N$ times the value $(N + 1)$ above, so that

$$2S_N = N(1 + N), \quad \text{so} \quad S_N = \frac{N(1 + N)}{2}.$$ 

Thus, the formula is confirmed.

**Example: Adding up the first 1000 integers**

Suppose we want to add up the first 1000 integers. This formula is very useful in what would otherwise be a huge calculation. We find that

$$S = 1 + 2 + 3 + \cdots + 1000 = \sum_{k=1}^{1000} k = \frac{1000(1 + 1000)}{2} = 500(1001) = 500500.$$ 

**Sums of squares and cubes**

We now present an argument verifying the above formulae for sums of squares and cubes.

First, note that

$$(k + 1)^3 - (k - 1)^3 = 6k^2 + 2,$$

so

$$\sum_{k=1}^{n} ((k + 1)^3 - (k - 1)^3) = \sum_{k=1}^{n} (6k^2 + 2).$$

But looking more carefully at the left hand side (LHS), we see that

$$\sum_{k=1}^{n} ((k + 1)^3 - (k - 1)^3) = 2^3 - 0^3 + 3^3 - 1^3 + 4^3 - 2^3 + 5^3 - 3^3 \cdots + (n + 1)^3 - (n - 1)^3$$

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*Contributed to these notes by Robert Israel.*
so most of the terms cancel, leaving only $-1 + n^3 + (n + 1)^3$. This means that

$$-1 + n^3 + (n + 1)^3 = 6 \sum_{k=1}^{n} k^2 + 2,$$

so

$$\sum_{k=1}^{n} k^2 = \frac{-1 + n^3 + (n + 1)^3 - 2n}{6} = \frac{2n^3 + 3n^2 + n}{6}.$$

Similarly, the formulae for $\sum_{k=1}^{n} k$ and $\sum_{k=1}^{n} k^3$, can be obtained by starting with the identities

$$(k + 1)^2 - (k - 1)^2 = 4k, \quad \text{and}\quad (k + 1)^4 - (k - 1)^4 = 8k^3 + 8k,$$

respectively. We encourage the interested reader to carry out these details.

An alternative approach using a technique called **mathematical induction** to verify the formulae for the sum of squares and cubes of integers is presented at the end of this chapter.

**Example: Volume of a Tower of Hanoi, revisited**

Armed with the formula for the sum of squares, we can now return to the problem of computing the volume of a tower of 100 stacked disks of heights 1 and radii $r = 1, 2, \ldots, 99, 100$. We have

$$V = \pi(1^2 + 2^2 + 3^2 + \cdots + 99^2 + 100^2) = \pi \sum_{k=1}^{100} k^2 = \pi \frac{100(101)(201)}{6} = 338,350\pi \text{ cubic units}.$$
Example

Compute the following sum:

\[ S_b = \sum_{k=10}^{50} k. \]

Solution

We can express the second sum as a difference of two sums:

\[ S_b = \sum_{k=10}^{50} k = \left( \sum_{k=1}^{50} k \right) - \left( \sum_{k=1}^{9} k \right). \]

Thus

\[ S_b = \left( \frac{50(51)}{2} - \frac{9(10)}{2} \right) = 1275 - 45 = 1230. \]

1.5 Summing the geometric series

Consider a sum of terms that all have the form \( r^k \), where \( r \) is some real number and \( k \) is an integer power. We refer to a series of this type as a geometric series. We have already seen one example of this type in a previous section. Below we will show that the sum of such a series is given by:

\[ S_N = 1 + r + r^2 + \ldots + r^N = \sum_{k=0}^{N} r^k = \frac{1 - r^{N+1}}{1 - r} \]  

where \( r \neq 1 \). We call this sum a (finite) geometric series. We would like to find an expression for terms of this form in the general case of any real number \( r \), and finite number of terms \( N \). First we note that there are \( N + 1 \) terms in this sum, so that if \( r = 1 \) then

\[ S_N = 1 + 1 + 1 + \ldots + 1 = N + 1 \]

(a total of \( N + 1 \) ones added.) If \( r \neq 1 \) we have the following trick:

\[ S_N = \frac{1 + r + r^2 + \ldots + r^N}{1} \]

\[ r S_N = \frac{r + r^2 + \ldots + r^N}{1} \]

Subtracting leads to

\[ S_N - r S_N = (1 + r + r^2 + \ldots + r^N) - (r + r^2 + \ldots + r^N + r^N) \]

Most of the terms on the right hand side cancel, leaving

\[ S_N(1 - r) = 1 - r^{N+1}. \]
Now dividing both sides by \(1 - r\) leads to

\[
S_N = \frac{1 - r^{N+1}}{1 - r},
\]

which was the formula to be established.

**Example: Geometric series**

Compute the following sum:

\[
S_c = \sum_{k=0}^{10} 2^k.
\]

**Solution**

This is a geometric series

\[
S_c = \sum_{k=0}^{10} 2^k = \frac{1 - 2^{10+1}}{1 - 2} = \frac{1 - 2048}{-1} = 2047.
\]

### 1.6 Prelude to infinite series

So far, we have looked at several examples of finite series, i.e. series in which there are only a finite number of terms, \(N\) (where \(N\) is some integer). We would like to investigate how the sum of a series behaves when more and more terms of the series are included. It is evident that in many cases, such as Gauss’s series (1.4), or sums of squared or cubed integers (e.g., Eqs. (1.2) and (1.3)), the series simply gets larger and larger as more terms are included. We say that such series diverge as \(N \to \infty\). Here we will look specifically for series that converge, i.e. have a finite sum, even as more and more terms are included\(^5\).

Let us focus again on the geometric series and determine its behaviour when the number of terms is increased. Our goal is to find a way of attaching a meaning to the expression

\[
S = \sum_{k=0}^{\infty} r^k,
\]

when the series becomes an infinite series. We will use the following definition:

### 1.6.1 The infinite geometric series

**Definition**

An infinite series that has a unique, finite sum is said to be *convergent*. Otherwise it is *divergent*.

---

\(^5\)Convergence and divergence of series is discussed in fuller depth in Chapter 11. However, these concepts are so important that some preliminary ideas need to be introduced early in the term.
1.6. Prelude to infinite series

Definition

Suppose that $S$ is an (infinite) series whose terms are $a_k$. Then the partial sums, $S_n$, of this series are

$$S_n = \sum_{k=0}^{n} a_k.$$ 

We say that the sum of the infinite series is $S$, and write

$$S = \sum_{k=0}^{\infty} a_k,$$ 

provided that

$$S = \lim_{n \to \infty} \sum_{k=0}^{n} a_k.$$ 

That is, we consider the infinite series as the limit of the partial sums as the number of terms $n$ is increased. In this case we also say that the infinite series converges to $S$.

We will see that only under certain circumstances will infinite series have a finite sum, and we will be interested in exploring two questions:

1. Under what circumstances does an infinite series have a finite sum.
2. What value does the partial sum approach as more and more terms are included.

In the case of a geometric series, the sum of the series, $(1.5)$ depends on the number of terms in the series, $n$ via $r^{n+1}$. Whenever $r > 1$, or $r < -1$, this term will get bigger in magnitude as $n$ increases, whereas, for $0 < r < 1$, this term decreases in magnitude with $n$. We can say that

$$\lim_{n \to \infty} r^{n+1} = 0 \quad \text{provided} \quad |r| < 1.$$ 

These observations are illustrated by two specific examples below. This leads to the following conclusion:

\begin{center}
The sum of an infinite geometric series,
\end{center}

$$S = 1 + r + r^2 + \cdots + r^k + \cdots = \sum_{k=0}^{\infty} r^k,$$

exists provided $|r| < 1$ and is

$$S = \frac{1}{1 - r}.$$ \hfill (1.6)

Examples of convergent and divergent geometric series are discussed below.
1.6.2 Example: A geometric series that converges.

Consider the geometric series with \( r = \frac{1}{2} \), i.e.

\[
S_n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots + \left(\frac{1}{2}\right)^n = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k.
\]

Then

\[
S_n = \frac{1 - (1/2)^{n+1}}{1 - (1/2)}.
\]

We observe that as \( n \) increases, i.e. as we retain more and more terms, we obtain

\[
\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - (1/2)^{n+1}}{1 - (1/2)} = \frac{1}{1 - (1/2)} = 2.
\]

In this case, we write

\[
\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots = 2
\]

and we say that “the (infinite) series converges to 2”.

1.6.3 Example: A geometric series that diverges

In contrast, we now investigate the case that \( r = 2 \): then the series consists of terms

\[
S_n = 1 + 2 + 2^2 + 2^3 + \ldots + 2^n = \sum_{k=0}^{n} 2^k = \frac{1 - 2^{n+1}}{1 - 2} = 2^{n+1} - 1
\]

We observe that as \( n \) grows larger, the sum continues to grow indefinitely. In this case, we say that the sum does not converge, or, equivalently, that the sum diverges.

It is important to remember that an infinite series, i.e. a sum with infinitely many terms added up, can exhibit either one of these two very different behaviours. It may converge in some cases, as the first example shows, or diverge (fail to converge) in other cases. We will see examples of each of these trends again. It is essential to be able to distinguish the two. Divergent series (or series that diverge under certain conditions) must be handled with particular care, for otherwise, we may find contradictions or seemingly reasonable calculations that have meaningless results.

1.7 Application of geometric series to the branching structure of the lungs

In this section, we will compute the volume and surface area of the branched airways of lungs\(^6\). We use the summation formulae to arrive at the results, and we also illustrate how the same calculation could be handled using a simple spreadsheet.

\(^6\)This section provides an example of how to set up a biologically relevant calculation based on geometric series. It is further studied in the homework problems. A similar example is given as an exercise for the student in Lab 1 of this calculus course.
1.7. Application of geometric series to the branching structure of the lungs

Our lungs pack an amazing large surface area into a confined volume. Most of the oxygen exchange takes place in tiny sacs called alveoli at the terminal branches of the airways passages. The bronchial tubes conduct air, and distribute it to the many smaller and smaller tubes that eventually lead to those alveoli. The principle of this efficient organ for oxygen exchange is that these very many small structures present a very large surface area. Oxygen from the air can diffuse across this area into the bloodstream very efficiently.

The lungs, and many other biological “distribution systems” are composed of a branched structure. The initial segment is quite large. It bifurcates into smaller segments, which then bifurcate further, and so on, resulting in a geometric expansion in the number of branches, their collective volume, length, etc. In this section, we apply geometric series to explore this branched structure of the lung. We will construct a simple mathematical model and explore its consequences. The model will consist in some well-formulated assumptions about the way that “daughter branches” are related to their “parent branch”. Based on these assumptions, and on tools developed in this chapter, we will then predict properties of the structure as a whole. We will be particularly interested in the volume $V$ and the surface area $S$ of the airway passages in the lungs.

![Figure 1.6. Air passages in the lungs consist of a branched structure. The index $n$ refers to the branch generation, starting from the initial segment, labeled 0. All segments are assumed to be cylindrical, with radius $r_n$ and length $\ell_n$ in the $n$'th generation.](image)

1.7.1 Assumptions

- The airway passages consist of many “generations” of branched segments. We label the largest segment with index “0”, and its daughter segments with index “1”, their successive daughters “2”, and so on down the structure from large to small branch segments. We assume that there are $M$ “generations”, i.e. the initial segment has undergone $M$ subdivisions. Figure 1.6 shows only generations 0, 1, and 2. (Typically, for human lungs there can be up to 25-30 generations of branching.)

- At each generation, every segment is approximated as a cylinder of radius $r_n$ and length $\ell_n$.

---

7The surface area of the bronchial tubes does not actually absorb much oxygen, in humans. However, as an example of summation, we will compute this area and compare how it grows to the growth of the volume from one branching layer to the next.
<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>radius of first segment</td>
<td>(r_0)</td>
<td>0.5 cm</td>
</tr>
<tr>
<td>length of first segment</td>
<td>(\ell_0)</td>
<td>5.6 cm</td>
</tr>
<tr>
<td>ratio of daughter to parent length</td>
<td>(\alpha)</td>
<td>0.9</td>
</tr>
<tr>
<td>ratio of daughter to parent radius</td>
<td>(\beta)</td>
<td>0.86</td>
</tr>
<tr>
<td>number of branch generations</td>
<td>(M)</td>
<td>30</td>
</tr>
<tr>
<td>average number daughters per parent</td>
<td>(b)</td>
<td>1.7</td>
</tr>
</tbody>
</table>

**Table 1.1. Typical structure of branched airway passages in lungs.**

- The number of branches grows along the “tree”. On average, each parent branch produces \(b\) daughter branches. In Figure 1.6, we have illustrated this idea for \(b = 2\). A branched structure in which each branch produces two daughter branches is described as a **bifurcating** tree structure (whereas **trifurcating** implies \(b = 3\)). In real lungs, the branching is slightly irregular. Not every level of the structure bifurcates, but in general, averaging over the many branches in the structure \(b\) is smaller than 2. In fact, the rule that links the number of branches in generation \(n\), here denoted \(x_n\), with the number (of smaller branches) in the next generation, \(x_{n+1}\) is

\[
x_{n+1} = bx_n.
\]

(1.7)

We will assume, for simplicity, that \(b\) is a constant. Since the number of branches is growing down the length of the structure, it must be true that \(b > 1\). For human lungs, on average, \(1 < b < 2\). Here we will take \(b\) to be constant, i.e. \(b = 1.7\). In actual fact, this simplification cannot be precise, because we have just one segment initially \((x_0 = 1)\), and at level 1, the number of branches \(x_1\) should be some small integer, not a number like “1.7”. However, as in many mathematical models, some accuracy is sacrificed to get intuition. Later on, details that were missed and are considered important can be corrected and refined.

- The ratios of radii and lengths of daughters to parents are approximated by “proportional scaling”. This means that the relationship of the radii and lengths satisfy simple rules: The lengths are related by

\[
\ell_{n+1} = \alpha \ell_n,
\]

(1.8)

and the radii are related by

\[
r_{n+1} = \beta r_n,
\]

(1.9)

with \(\alpha\) and \(\beta\) positive constants. For example, it could be the case that the radius of daughter branches is 1/2 or 2/3 that of the parent branch. Since the branches decrease in size (while their number grows), we expect that \(0 < \alpha < 1\) and \(0 < \beta < 1\).

Rules such as those given by equations (1.8) and (1.9) are often called **self-similar growth** laws. Such concepts are closely linked to the idea of fractals, i.e. theoretical structures produced by iterating such growth laws indefinitely. In a real biological structure, the
number of generations is finite. (However, in some cases, a finite geometric series is well-
approximated by an infinite sum.)

Actual lungs are not fully symmetric branching structures, but the above approxima-
tions are used here for simplicity. According to physiological measurements, the scale
factors for sizes of daughter to parent size are in the range \(0.65 \leq \alpha, \beta \leq 0.9\). (K. G.
the purposes of this example, we will use the values of constants given in Table 1.1.

### 1.7.2 A simple geometric rule

The three equations that govern the rules for successive branching, i.e. equations (1.7), (1.8),
and (1.9), are examples of a very generic “geometric progression” recipe. Before returning
to the problem at hand, let us examine the implications of this recursive rule, when it is
applied to generating the whole structure. Essentially, we will see that the rule linking two
generations implies an exponential growth. To see this, let us write out a few first terms in
the progression of the sequence \(\{x_n\}\):

- *initial value:* \(x_0\)
- *first iteration:* \(x_1 = bx_0\)
- *second iteration:* \(x_2 = bx_1 = b(bx_0) = b^2x_0\)
- *third iteration:* \(x_3 = bx_2 = b(b^2x_0) = b^3x_0\)

... 

By the same pattern, at the \(n\)’th generation, the number of segments will be

\[n\text{’th iteration: } x_n = bx_{n-1} = b(bx_{n-2}) = b(b(bx_{n-3})) = \cdots = (b \cdot b \cdots b) x_0 = b^n x_0.\]

We have arrived at a simple, but important result, namely:

| The rule linking two generations, \(x_n = bx_{n-1}\) (1.10) |
| \(x_n = b^n x_0.\) (1.11) |

This connection between the rule linking two generations and the resulting number of
members at each generation is useful in other circumstances. Equation (1.10) is sometimes
called a *recursion relation*, and its solution is given by equation (1.11). We will use the
same idea to find the connection between the volumes, and surface areas of successive
segments in the branching structure.
1.7.3 Total number of segments

We used the result of Section 1.7.2 and the fact that there is one segment in the 0’th generation, i.e. \( x_0 = 1 \), to conclude that at the \( n \)'th generation, the number of segments is

\[
x_n = x_0 b^n = 1 \cdot b^n = b^n.
\]

For example, if \( b = 2 \), the number of segments grows by powers of 2, so that the tree bifurcates with the pattern 1, 2, 4, 8, etc.

To determine how many branch segments there are in total, we add up over all generations, \( 0, 1, \ldots, M \). This is a geometric series, whose sum we can compute. Using equation (1.5), we find

\[
N = \sum_{n=0}^{M} b^n = \left( \frac{1 - b^{M+1}}{1 - b} \right).
\]

Given \( b \) and \( M \), we can then predict the exact number of segments in the structure. The calculation is summarized further on for values of the branching parameter, \( b \), and the number of branch generations, \( M \), given in Table 1.1.

1.7.4 Total volume of airways in the lung

Since each lung segment is assumed to be cylindrical, its volume is

\[
v_n = \pi r_n^2 \ell_n.
\]

Here we mean just a single segment in the \( n \)'th generation of branches. (There are \( b^n \) such identical segments in the \( n \)'th generation, and we will refer to the volume of all of them together as \( V_n \) below.)

The length and radius of segments also follow a geometric progression. In fact, the same idea developed above can be used to relate the length and radius of a segment in the \( n \)'th generation to the length and radius of the original 0’th generation segment, namely,

\[
\ell_n = \alpha \ell_{n-1} \Rightarrow \ell_n = \alpha^n \ell_0,
\]

and

\[
r_n = \beta r_{n-1} \Rightarrow r_n = \beta^n r_0.
\]

Thus the volume of one segment in generation \( n \) is

\[
v_n = \pi r_n^2 \ell_n = \pi (\beta^n r_0)^2 (\alpha^n \ell_0) = (\alpha \beta^2)^n \left( \pi r_0^2 \ell_0 \right).
\]

This is just a product of the initial segment volume \( v_0 = \pi r_0^2 \ell_0 \), with the \( n \)'th power of a certain factor(\( \alpha, \beta \)). (That factor takes into account that both the radius and the length are being scaled down at every successive generation of branching.) Thus

\[
v_n = (\alpha \beta^2)^n v_0.
\]
The total volume of all \((b^n)\) segments in the \(n\)'th layer is
\[
V_n = b^n v_n = b^n (\alpha \beta^2)^n v_0 = \left(\frac{b\alpha\beta^2}{a}\right)^n v_0.
\]

Here we have grouped terms together to reveal the simple structure of the relationship: one part of the expression is just the initial segment volume, while the other is now a “scale factor” that includes not only changes in length and radius, but also in the number of branches. Letting the constant \(a\) stand for that scale factor, \(a = (b\alpha\beta^2)\) leads to the result that the volume of all segments in the \(n\)'th layer is
\[
V_n = a^n v_0.
\]

The total volume of the structure is obtained by summing the volumes obtained at each layer. Since this is a geometric series, we can use the summation formula, i.e., Equation (1.5). Accordingly, total airways volume is
\[
V = \sum_{n=0}^{30} V_n = v_0 \sum_{n=0}^{30} a^n = v_0 \left(1 - a^{M+1}\right) \left(1 - a\right). 
\]

The similarity of treatment with the previous calculation of number of branches is apparent. We compute the value of the constant \(a\) in Table 1.2, and find the total volume in Section 1.7.6.

### 1.7.5 Total surface area of the lung branches

The surface area of a single segment at generation \(n\), based on its cylindrical shape, is
\[
s_n = 2\pi r_n \ell_n = 2\pi (\beta^n r_0)(\alpha^n \ell_0) = (\alpha \beta)^n \left(2\pi r_0 \ell_0\right) \frac{s_0}{s_0},
\]
where \(s_0\) is the surface area of the initial segment. Since there are \(b^n\) branches at generation \(n\), the total surface area of all the \(n\)'th generation branches is thus
\[
S_n = b^n (\alpha \beta)^n s_0 = \left(\frac{b\alpha\beta}{c}\right)^n s_0,
\]
where we have let \(c\) stand for the scale factor \(c = (b\alpha\beta)\). Thus,
\[
S_n = c^n s_0.
\]

This reveals the similar nature of the problem. To find the total surface area of the airways, we sum up,
\[
S = s_0 \sum_{n=0}^{M} c^n = s_0 \left(1 - c^{M+1}\right) \left(1 - c\right). 
\]

We compute the values of \(s_0\) and \(c\) in Table 1.2, and summarize final calculations of the total airways surface area in Section 1.7.6.
volume of first segment $v_0 = \pi r_0^2 \ell_0$ 4.4 cm$^3$
surface area of first segment $s_0 = 2\pi r_0 \ell_0$ 17.6 cm$^2$
ratio of daughter to parent segment volume $(\alpha \beta^2)$ 0.66564
ratio of daughter to parent segment surface area $(\alpha \beta)$ 0.774
ratio of net volumes in successive generations $a = b \alpha \beta^2$ 1.131588
ratio of net surface areas in successive generations $c = b \alpha \beta$ 1.3158

Table 1.2. Volume, surface area, scale factors, and other derived quantities. Because $a$ and $c$ are bases that will be raised to large powers, it is important to that their values are fairly accurate, so we keep more significant figures.

1.7.6 Summary of predictions for specific parameter values

By setting up the model in the above way, we have revealed that each quantity in the structure obeys a simple geometric series, but with distinct “bases” $b, a$ and $c$ and coefficients 1, $v_0$, and $s_0$. This approach has shown that the formula for geometric series applies in each case. Now it remains to merely “plug in” the appropriate quantities. In this section, we collect our results, use the sample values for a model “human lung” given in Table 1.1, or the resulting derived scale factors and quantities in Table 1.2 to finish the task at hand.

Total number of segments

$$N = \sum_{n=0}^{M} b^n = \left( \frac{1 - b^{M+1}}{1 - b} \right) = \left( \frac{1 - (1.7)^{31}}{1 - 1.7} \right) = 1.9898 \cdot 10^7 \approx 2 \cdot 10^7.$$  

According to this calculation, there are a total of about 20 million branch segments overall (including all layers, from top to bottom) in the entire structure!

Total volume of airways

Using the values for $a$ and $v_0$ computed in Table 1.2, we find that the total volume of all segments is

$$V = v_0 \sum_{n=0}^{30} a^n = v_0 \left( \frac{1 - a^{M+1}}{1 - a} \right) = 4.4 \left( \frac{1 - 1.131588^{31}}{1 - 1.131588} \right) = 1510.3 \text{ cm}^3.$$  

Recall that 1 litre = 1000 cm$^3$. Then we have found that the lung airways contain about 1.5 litres.

Total surface area of airways

Using the values of $s_0$ and $c$ in Table 1.2, the total surface area of the tubes that make up the airways is

$$S = s_0 \sum_{n=0}^{M} c^n = s_0 \left( \frac{1 - c^{M+1}}{1 - c} \right) = 17.6 \left( \frac{1 - 1.3158^{31}}{1 - 1.3158} \right) = 2.76 \cdot 10^5 \text{ cm}^2.$$
There are 100 cm per meter, and $(100)^2 = 10^4$ cm$^2$ per m$^2$. Thus, the area we have computed is equivalent to about 28 square meters!

### 1.7.7 Exploring the problem numerically

Up to now, all calculations were done using the formulae developed for geometric series. However, sometimes it is more convenient to devise a computer algorithm to implement “rules” and perform repetitive calculations in a problem such as discussed here. The advantage of that approach is that it eliminates tedious calculations by hand, and, in cases where summation formulae are not know to us, reduces the need for analytical computations. It can also provide a shortcut to visual summary of the results. The disadvantage is that it can be less obvious how each of the values of parameters assigned to the problem affects the final answers.

A spreadsheet is an ideal tool for exploring iterated rules such as those given in the lung branching problem. In Figure 1.7 we show the volumes and surface areas associated with the lung airways for parameter values discussed above. Both layer by layer values and cumulative sums leading to total volume and surface area are shown in each of (a) and (c). In (b) and (d), we compare these results to similar graphs in the case that one parameter, the branching number, $b$ is adjusted from 1.7 (original value) to 2. The contrast between the graphs shows how such a small change in this parameter can significantly affect the results.

### 1.7.8 For further independent study

The following problems can be used for further independent exploration of these ideas.

1. In our model, we have assumed that, on average, a parent branch has only “1.7” daughter branches, i.e. that $b = 1.7$. Suppose we had assumed that $b = 2$. What would the total volume $V$ be in that case, keeping all other parameters the same? Explain why this is biologically impossible in the case $M = 30$ generations. For what value of $M$ would $b = 2$ lead to a reasonable result?

2. Suppose that the first 5 generations of branching produce 2 daughters each, but then from generation 6 on, the branching number is $b = 1.7$. How would you set up this variant of the model? How would this affect the calculated volume?

3. In the problem we explored, the net volume and surface area keep growing by larger and larger increments at each “generation” of branching. We would describe this as “unbounded growth”. Explain why this is the case, paying particular attention to the scale factors $a$ and $c$.

4. Suppose we want a set of tubes with a large surface area but small total volume. Which single factor or parameter should we change (and how should we change it) to correct this feature of the model, i.e. to predict that the total volume of the branching tubes remains roughly constant while the surface area increases as branching layers are added.

---

*See Lab 1 for a similar problem that is also investigated using a spreadsheet.*
Figure 1.7. (a) $V_n$, the volume of layer $n$ (red bars), and the cumulative volume down to layer $n$ (yellow bars) are shown for parameters given in Table 1.1. (b) Same as (a) but assuming that parent segments always produce two daughter branches (i.e. $b = 2$). The graphs in (a) and (b) are shown on the same scale to accentuate the much more dramatic growth in (b). (c) and (d): same idea showing the surface area of $n$th layer (green) and the cumulative surface area to layer $n$ (blue) for original parameters (in c), as well as for the value $b = 2$ (in d).

5. Determine how the branching properties of real human lungs differs from our assumed model, and use similar ideas to refine and correct our estimates. You may want to investigate what is known about the actual branching parameter $b$, the number of generations of branches, $M$, and the ratios of lengths and radii that we have assumed. Alternately, you may wish to find parameters for other species and do a comparative study of lungs in a variety of animal sizes.

6. Branching structures are ubiquitous in biology. Many species of plants are based on a regular geometric sequence of branching. Consider a tree that trifurcates (i.e. produces 3 new daughter branches per parent branch, $b = 3$). Explain: (a) What biological problem is to be solved in creating such a structure (b) What sorts of
1.8 Summary

In this chapter, we collected useful formulae for areas and volumes of simple 2D and 3D shapes. A summary of the most important ones is given below. Table 1.3 lists the areas of simple shapes, Table 1.4 the volumes and Table 1.5 the surface areas of 3D shapes.

We used areas of triangles to compute areas of more complicated shapes, including regular polygons. We used a polygon with \( N \) sides to approximate the area of a circle, and then, by letting \( N \) go to infinity, we were able to prove that the area of a circle of radius \( r \) is \( A = \pi r^2 \). This idea, and others related to it, will form a deep underlying theme in the next two chapters and later on in this course.

We introduced some notation for series and collected useful formulae for summation of such series. These are summarized in Table 1.6. We will use these extensively in our next chapter.

Finally, we investigated geometric series and studied a biological application, namely the branching structure of lungs.

<table>
<thead>
<tr>
<th>Object</th>
<th>dimensions</th>
<th>area, ( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>triangle</td>
<td>base ( b ), height ( h )</td>
<td>( \frac{1}{2}bh )</td>
</tr>
<tr>
<td>rectangle</td>
<td>base ( b ), height ( h )</td>
<td>( bh )</td>
</tr>
<tr>
<td>circle</td>
<td>radius ( r )</td>
<td>( \pi r^2 )</td>
</tr>
</tbody>
</table>

Table 1.3. Areas of planar regions
### Table 1.4. Volumes of 3D shapes

<table>
<thead>
<tr>
<th>Object</th>
<th>dimensions</th>
<th>volume, $V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>box</td>
<td>base $b$, height $h$, width $w$</td>
<td>$hwb$</td>
</tr>
<tr>
<td>circular cylinder</td>
<td>radius $r$, height $h$</td>
<td>$\pi r^2 h$</td>
</tr>
<tr>
<td>sphere</td>
<td>radius $r$</td>
<td>$\frac{4}{3} \pi r^3$</td>
</tr>
<tr>
<td>cylindrical shell*</td>
<td>radius $r$, height $h$, thickness $\tau$</td>
<td>$2\pi rh\tau$</td>
</tr>
<tr>
<td>spherical shell*</td>
<td>radius $r$, thickness $\tau$</td>
<td>$4\pi r^2 \tau$</td>
</tr>
</tbody>
</table>

*Assumes a thin shell, i.e. small $\tau$.

### Table 1.5. Surface areas of 3D shapes

<table>
<thead>
<tr>
<th>Object</th>
<th>dimensions</th>
<th>surface area, $S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>box</td>
<td>base $b$, height $h$, width $w$</td>
<td>$2(bh + bw + hw)$</td>
</tr>
<tr>
<td>circular cylinder</td>
<td>radius $r$, height $h$</td>
<td>$2\pi rh$</td>
</tr>
<tr>
<td>sphere</td>
<td>radius $r$</td>
<td>$4\pi r^2$</td>
</tr>
</tbody>
</table>

### Table 1.6. Useful summation formulae.

<table>
<thead>
<tr>
<th>Sum</th>
<th>Notation</th>
<th>Formula</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 + 2 + 3 + \cdots + N$</td>
<td>$\sum_{k=1}^{N} k$</td>
<td>$\frac{N(N+1)}{2}$</td>
<td>Gauss’ formula</td>
</tr>
<tr>
<td>$1^2 + 2^2 + 3^2 + \cdots + N^2$</td>
<td>$\sum_{k=1}^{N} k^2$</td>
<td>$\frac{N(N+1)(2N+1)}{6}$</td>
<td>Sum of squares</td>
</tr>
<tr>
<td>$1^3 + 2^3 + 3^3 + \cdots + N^3$</td>
<td>$\sum_{k=1}^{N} k^3$</td>
<td>$\left(\frac{N(N+1)}{2}\right)^2$</td>
<td>Sum of cubes</td>
</tr>
<tr>
<td>$1 + r + r^2 + r^3 \ldots r^N$</td>
<td>$\sum_{k=0}^{N} r^k$</td>
<td>$\frac{1-r^{N+1}}{1-r}$</td>
<td>Geometric sum</td>
</tr>
</tbody>
</table>
1.9 Optional Material

1.9.1 How to prove the formulae for sums of squares and cubes

Mathematicians are concerned with rigorously establishing formulae such as sums of squared (or cubed) integers. While it is not hard to see that these formulae “work” for a few cases, determining that they work in general requires more work. Here we provide a taste of how such careful arguments works. We give two examples. The first, based on mathematical induction provides a general method that could be used in many similar kinds of proofs. The second argument, also for purposes of illustration uses a “trick”. Devising such tricks is not as straightforward, and depends to some degree on serendipity or experience with numbers.

Proof by induction

Here, we prove the formula for the sum of square integers,

$$\sum_{k=1}^{N} k^2 = \frac{N(N + 1)(2N + 1)}{6},$$

using a technique called induction. The idea of the method is to check that the formula works for one or two simple cases (e.g. the “sum” of just one or just two terms of the series), and then show that whenever it works for one case (summing up to $N$), it has to also work for the next case (summing up to $N + 1$).

First, we verify that this formula works for a few test cases:

$N = 1$: If there is only one term, then clearly, by inspection,

$$\sum_{k=1}^{1} k^2 = 1^2 = 1.$$

The formula indicates that we should get

$$S = \frac{1(1 + 1)(2 \cdot 1 + 1)}{6} = \frac{1(2)(3)}{6} = 1,$$

so this case agrees with the prediction.

$N = 2$:

$$\sum_{k=1}^{2} k^2 = 1^2 + 2^2 = 1 + 4 = 5.$$

The formula would then predict that

$$S = \frac{2(2 + 1)(2 \cdot 2 + 1)}{6} = \frac{2(3)(5)}{6} = 5.$$

So far, elementary computation matches with the result predicted by the formula. Now we show that if this formula holds for any one case, e.g. for the sum of the first $N$
squares, then it is also true for the next case, i.e. for the sum of $N + 1$ squares. So we will assume that someone has checked that for some particular value of $N$ it is true that

$$S_N = \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}.$$

Now the sum of the first $N + 1$ squares will be just a bit bigger: it will have one more term added to it:

$$S_{N+1} = \sum_{k=1}^{N+1} k^2 = \sum_{k=1}^{N} k^2 + (N+1)^2.$$

Thus

$$S_{N+1} = \frac{N(N+1)(2N+1)}{6} + (N+1)^2.$$

Combining terms, we get

$$S_{N+1} = (N+1)\left[\frac{N(2N+1)}{6} + (N+1)\right],$$

$$S_{N+1} = (N+1)\frac{2N^2 + N + 6N + 6}{6} = (N+1)\frac{2N^2 + 7N + 6}{6}.$$

Simplifying and factoring the last term leads to

$$S_{N+1} = (N+1)\frac{(2N+3)(N+2)}{6}.$$

We want to check that this still agrees with what the formula predicts. To make the notation simpler, we will let $M$ stand for $N + 1$. Then, expressing the result in terms of the quantity $M = N + 1$ we get

$$S_M = \sum_{k=1}^{M} k^2 = (N+1)\frac{[2(N+1) + 1][(N+1) + 1]}{6} = M\frac{[2M + 1][M + 1]}{6}.$$

This is the same formula as we started with, only written in terms of $M$ instead of $N$. Thus we have verified that the formula works. By Mathematical Induction we find that the result has been proved.
1.10 Exercises

Exercise 1.1 Answer the following questions:

(a) What is the value of the fifth term of the sum \( S = \sum_{k=1}^{20} \frac{(5 + 3k)}{k} \)?

(b) How many terms are there in total in the sum \( S = \sum_{k=7}^{17} e^k \)?

(c) Write out the terms in \( \sum_{n=1}^{5} 2^{n-1} \).

(d) Write out the terms in \( \sum_{n=0}^{4} 2^n \).

(e) Write the series \( 1 + 3 + 3^2 + 3^3 \) in summation notation in two equivalent forms.

Exercise 1.2 Summation notation

(a) Write \( 2 + 4 + 6 + 8 + 10 + 12 + \ldots \) in summation notation.

(b) Write \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \) in summation notation.

(c) Write out the first few terms of \( \sum_{i=0}^{100} 3^i \)

(d) Write out the first few terms of \( \sum_{n=1}^{\infty} \frac{1}{n^n} \)

(e) Simplify \( \sum_{k=5}^{\infty} \left( \frac{1}{2} \right)^k + \sum_{k=2}^{4} \left( \frac{1}{2} \right)^k \)

(f) Simplify \( \sum_{x=0}^{50} 3^x - \sum_{x=10}^{50} 3^x \)

(g) Simplify \( \sum_{n=0}^{100} n + \sum_{n=0}^{100} n^2 \)

(h) Simplify \( \sum_{y=0}^{100} y + \sum_{y=0}^{100} y^2 + \sum_{y=0}^{100} 1 \)

Exercise 1.3 Show that the following pairs of sums are equivalent:

(a) \( \sum_{m=0}^{10} (m + 1)^2 \) and \( \sum_{n=1}^{11} n^2 \)
(b) \[ \sum_{n=1}^{4} (n^2 - 2n + 1) \quad \text{and} \quad \sum_{n=1}^{4} (n - 1)^2 \]

**Exercise 1.4** Compute the following sums: (Some of the calculations will be facilitated by a calculator)

(a) \[ \sum_{i=1}^{290} 1 \]
(b) \[ \sum_{n=1}^{150} 2 \]
(c) \[ \sum_{i=1}^{80} 3 \]
(d) \[ \sum_{n=1}^{50} n \]
(e) \[ \sum_{n=1}^{60} n \]
(f) \[ \sum_{n=1}^{60} n \]
(g) \[ \sum_{n=20}^{100} n \]
(h) \[ \sum_{n=1}^{25} 3n^2 \]
(i) \[ \sum_{n=1}^{20} 2n^2 \]
(j) \[ \sum_{i=1}^{75} (i + 2) \]
(k) \[ \sum_{i=1}^{75} (i + 1) \]
(l) \[ \sum_{k=100}^{500} k \]
(m) \[ \sum_{k=50}^{100} k \]
(n) \[ \sum_{k=2}^{50} (k^2 - 2k + 1) \]
(o) \[ \sum_{k=5}^{50} (k^2 - 2k + 1) \]
(p) \[ \sum_{m=10}^{20} m^3 \]
(q) \[ \sum_{m=0}^{15} (m + 1)^2 \]

For the solutions to these, we will use several summation formulae, and the notation shown below for convenience:

\[ S_0(n) = \sum_{i=1}^{n} 1 = n \]
\[ S_1(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]
\[ S_2(n) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \]
\[ S_3(n) = \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 \]

**Exercise 1.5** Use the sigma summation notation to set up the following problems, and then apply known formulae to compute the sums.

(a) Find the sum of the first 50 even numbers, \( 2 + 4 + 6 + ... \)

(b) Find the sum of the first 50 odd numbers, \( 1 + 3 + ... \)

(c) Find the sum of the first 50 integers of the form \( n(n + 1) \) where \( n = 1, 2, ..., 50 \).

(d) Consider all the integers that are of the form \( n(n - 1) \) where \( n = 1, 2, 3, ... \). Find the sum of the first 50 such numbers.

**Exercise 1.6** Compute the following sum.

\[ S = \sum_{i=1}^{12} i(1 - i) + 2^i \]
**Exercise 1.7** A clock at London’s Heathrow airport chimes every half hour. At the beginning of the \( n \)’th hour, the clock chimes \( n \) times. (For example, at 8:00 AM the clock chimes 8 times, at 2:00 PM the clock chimes fourteen times, and at midnight the clock chimes 24 times.) The clock also chimes once at half-past every hour. Determine how many times in total the clock chimes in one full day. Use sigma notation to write the form of the series, and then find its sum.

**Exercise 1.8** A set of Japanese lacquer boxes have been made to fit one inside the other. All the boxes are cubical, and they have sides of lengths 1, 2, 3...15 inches. Find the total volume enclosed by all the boxes combined. Ignore the thickness of the walls of the boxes. (Calculator may be helpful.)

**Exercise 1.9** A framing shop uses a square piece of matt cardboard to create a set of square frames, one cut out from the other, with as little wasted as possible. The original piece of cardboard is 50 cm by 50 cm. Each of the “nested” square frames (see Problem 1.8 for the definition of nested) has a border 2 cm thick. How many frames in all can be made from this original piece of cardboard? What is the total area that can be enclosed by all these frames together?

![Figure 1.8. For problem 1.9](image)

**Exercise 1.10** The Great Pyramid of Giza, Egypt, built around 2,720-2,560 BC by Khufu (also known as Cheops) has a square base. We will assume that the base has side length 200 meters. The pyramid is made out of blocks of stone whose size is roughly \( 1 \times 1 \times 0.73 \) m\(^3\). There are 200 layers of blocks, so that the height of the pyramid is \( 200 \times 0.73 = 146 \) m. Assume that the size of the pyramid steps (i.e. the horizontal distance between the end of one step and the beginning of another) is 0.5 m. We will also assume that the pyramid is solid, i.e. we will neglect the (relatively small) spaces that make up passages and burial chambers inside the structure.

(a) How many blocks are there in the layer that makes up the base of the pyramid? How many blocks in the second layer?

(b) How many blocks are there at the very top of the pyramid?

(c) Write down a summation formula for the total number of blocks in the pyramid and compute the total. (Hint: you may find it easiest to start the sum from the top layer and
Chapter 1. Areas, volumes and simple sums

Exercises

Exercise 1.11 Your local produce store has a special on oranges. Their display of fruit is a triangular pyramid with 100 layers, topped with a single orange (i.e. top layer: 1). The layer second from the top has three ($3 = 1 + 2$) oranges, and the one directly under it has six ($6 = 3 + 2 + 1$). The same pattern continues for all 100 layers. (This results in efficient “hexagonal” packing, with each orange sitting in a little depression created by three neighbours right under it.)

(a) How many oranges are there in the fourth and fifth layers from the top? How many in the $N^{th}$ layer from the top?

(b) If the “pyramid of oranges” only has 3 layers, how many oranges are used in total? What if the pyramid has 4, or 5 layers?

(c) Write down a formula for the sum of the total number of oranges that would be needed to make a pyramid with $N$ layers. Simplify your result so that you can use the summation formulae for $\sum n$ and for $\sum n^2$ to determine the total number of oranges in such a pyramid.

(d) Determine how many oranges are needed for the pyramid with 100 layers.

Exercise 1.12 A right circular cone (shaped like “an inverted ice cream cone”), has height $h$ and a circular base (radius $r$) which is perpendicular to the cone’s axis. In this exercise you will calculate the volume of this cone.

Figure 1.9. For problem 1.12. The cone with approximating $N$ disks.

(1) Make $N$ uniform slices of the cone, each one parallel to the bottom, and of height $h/N$. Inside each slice put a cylindrical disk of the same height. The radius of the slices vary from 0 at the top to nearly $r$ at the bottom. (See Figure 1.9, $N = 10$.) Use similar triangles to answer these questions:
(a) What is the smallest disk radius other than 0?
(b) What is the radius of the kth disk?

(2) Express the total volume of the N disks as a sum and evaluate it.

(3) As N gets larger, what is the limit of this sum? (This is the volume of the cone.)

**Exercise 1.13** A (finite) geometric series with k terms is a sum of the form

\[ 1 + r + r^2 + r^3 + \ldots + r^{k-1} = \sum_{n=0}^{k-1} r^n \]

and is given by the formula

\[ S = \frac{1 - r^k}{1 - r} \quad \text{provided} \quad r \neq 1. \]

(a) This formula does not work if \( r = 1 \). Find the actual value of the series for \( r = 1 \).

(b) Express in summation notation and find the sum of the series \( 1 + 2^1 + 2^2 + 2^3 + \ldots + 2^{10} \).

(c) Express in summation notation and find the sum of the series \( 1 + (0.5)^1 + (0.5)^2 + (0.5)^3 + \ldots + (0.5)^{10} \).

**Exercise 1.14** Use the sum of a geometric series to answer this question (use a calculator).

(a) Find the sum of the first 11 numbers of the form \( 1.1^k \) for \( k = 0, 1, 2, \ldots \). Now find the sum of the first 21 such numbers, the first 31 such numbers, the first 41 such numbers, and the first 51 such numbers.

(b) Repeat the process but now find sums of the numbers \( 0.9^k, \quad k = 0, 1, 2 \ldots \)

(c) What do you notice about the pattern of results in part (a) and in part (b)? Can you explain what is happening in each of these cases and why they are different?

(d) Now consider the general problem of finding a value for the sum

\[ \sum_{k=0}^{N} r^k \]

when the number \( N \) gets larger and larger. Suggest under what circumstances this sum will stay finite, and what value that finite sum will approach. To do this, you should think about the formula for the finite geometric sum and determine how it behaves for various values of \( r \) as \( N \) gets very large. This idea will be very important when we discuss infinite series.
Exercise 1.15  According to legend, the inventor of the game of chess (in Persia) was offered a prize for his clever invention. He requested payment in kind, i.e. in kernels of grain. He asked to be paid $1$ kernel for the first square of the board, two for the second, four for the third, etc. Use a summation formula to determine the total number of kernels of grain he would have earned in total. (Hint: a chess board has $8 \times 8 = 64$ squares and the first square contains $2^0 = 1$ kernel.)

Exercise 1.16  A branching colony of fungus starts as a single spore with a single segment of filament growing out of it. This will be called generation $0$. The tip of the filament branches, producing two new segments. Each tip then branches again and the process repeats. Suppose there have been $10$ such branching events. How many tips will there be? If each segment is the same length ($1$ unit), what will be the total length of all the segments combined after $10$ branching events? (Include the length of the initial single segment in your answer.)

Exercise 1.17  A branching plant and geometric series A plant grows by branching, starting with one segment of length $\ell_0$ (in the $0$'th generation). Every parent branch has exactly two daughter branches. The length of each daughter branch is $(2/5)$ times the length of the parent branch. (Your answers will be in terms of $\ell_0$.)

(a) Find the total length of just the $12^{th}$ generation branch segments.

(b) Find the total length of the whole structure including the original segment and all $12$ successive generations.

(c) Find the approximate total length of all segments in the whole structure if the plant keeps on branching forever.

Exercise 1.18  Branching airways, continued Consider the branching airways in the lungs. Suppose that the initial bronchial segment has length $\ell_0$ and radius $r_0$. Let $\alpha$ and $\beta$ be the scale factors for the length and radius, respectively, of daughter branches (i.e. in a branching event, assume that $\ell_{n+1} = \alpha \ell_n$ and $r_{n+1} = \beta r_n$ are the relations that link daughters to parent branches, and that $0 < \alpha < 1$, $0 < \beta < 1$, $l_i > 0$, $r_i > 0$ for all $i$). Let $b$ be the average number of daughters per parent branch. Let $F_n = S_n/V_n$ be the ratio of total surface area to total volume in the $n$th layer of the structure (i.e. for the $n$th generation branches).

(a) Find $F_n$ in terms of $\ell_0$, $r_0$, $b$, $\beta$, $\alpha$.

(b) In the lungs, it would be reasonable to expect that the surface area to volume ratio should increase from the initial segment down through the layers. What should be true of the parameters for this to be the case?

Exercise 1.19  Branching lungs

(a) Consider branched airways that have the following geometric properties (Table a). Find the total number of branch segments, the volume and the surface area of this branched structure. Note that a calculator will be helpful.
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<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>radius of first segment</td>
<td>$r_0$</td>
</tr>
<tr>
<td>length of first segment</td>
<td>$\ell_0$</td>
</tr>
<tr>
<td>ratio of daughter to parent length</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>ratio of daughter to parent radius</td>
<td>$\beta$</td>
</tr>
<tr>
<td>number of branch generations</td>
<td>$M$</td>
</tr>
<tr>
<td>average number daughters per parent</td>
<td>$b$</td>
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</table>

(b) What happens as $M$ gets larger? Will the volume and the surface area approach some finite limit, or will they grow indefinitely? How should the parameter $\beta$ be changed so that the surface area will keep increasing while the volume stays finite as $M$ increases?

**Exercise 1.20  Using simple geometry to compute an area**

(a) Find the area of a regular octagon (a polygon that has eight equal sides). Assume that the length of each side is 1 cm.

(b) What is the area of the smallest circle that can be drawn around this octagon?
1.11 Solutions

Solution to 1.1

(a) \( a_5 = 4 \)  
(b) 11 terms  
(c) \( 1 + 2 + 2^2 + 2^3 + 2^4 \)  
(d) \( 1 + 2 + 2^2 + 2^3 + 2^4 \)  
(e) \( \sum_{n=0}^{3} 3^n \)

Solution to 1.2

(a) \( \sum_{n=1}^{\infty} 2n \)  
(b) \( \sum_{n=1}^{\infty} \frac{1}{n} \)  
(c) \( 1 + 3 + 9 + 27 + 81 + 243 + \ldots \)  
(d) \( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \)  
(e) \( \sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^k \)  
(f) \( \sum_{x=0}^{9} 3^x \)  
(g) \( \sum_{n=0}^{100} (n + n^2) \)  
(h) \( \sum_{y=0}^{100} (y + 1)^2 \)

Solution to 1.3

(a) each sum represents \( 1 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots + 11^2 \)  
(b) each sum represents \( 0 + 1 + 4 + 9 \)

Solution to 1.4

(a) 290  
(b) 300  
(c) 240  
(d) 1275  
(e) 1830  
(f) 1785  
(g) 4860  
(h) 16575  
(i) 5740  
(j) 1650  
(k) 2925  
(l) 120300  
(m) 3825  
(n) 40425  
(o) 40411  
(p) 42075  
(q) 18496

Solution to 1.5

(a) 2550  
(b) 2500  
(c) 44200  
(d) 41650

Solution to 1.6 \( S = 7618 \)

Solution to 1.7 The clock chimes 324 times.

Solution to 1.8 The total enclosed volume is 14,400 cubic inches.
Solution to 1.9  12 frames can be made enclosing a total of 9200 cm$^2$.

Solution to 1.10  
(a) 39601  
(b) 1  
(c) 2,686,700

Solution to 1.11  
(a) $\frac{k(k + 1)}{2}$  
(b) 10, 20, 35  
(c) $\frac{N(N + 1)(N + 2)}{6}$  
(d) 171700

Solution to 1.12  
(1) (a) The radius of the smallest disk is $r_1 = \frac{r}{N}$.
   
   (b) The radius of the $k$-th disk is $r_k = \frac{kr}{N}$.
   
   (2) $V = \pi \sum_{k=1}^{N} \left( \frac{kr}{N} \right)^2 \frac{h}{N} = \pi r^2 h \frac{2N^2 + 3N + 1}{6N^2}$.
   
   (3) The volume converges to $V \to \frac{1}{3} \pi r^2 h$.

Solution to 1.13  
(a) $k$  
(b) 2047  
(c) $2 \left( 1 - \frac{1}{2^{11}} \right)$

Solution to 1.14  
(a) Approximately 18.531, 64.002, 181.943, 487.852, and 1281.299, respectively.

(b) Approximately 6.862, 8.906, 9.618, 9.867, and 9.954, respectively.

(c) In (a), for $|r| > 1$, the sums increase (rapidly) with $N$ whereas in (b), for $|r| < 1$, they increase only very slowly.

(d) For $N \to \infty$, the sum converges for $|r| < 1$ and diverges otherwise.

Solution to 1.15  If there had been enough grain on Earth, the inventor would have received $2^{64} - 1 \approx 1.845 \times 10^{19}$ grains.

Solution to 1.16  The total length of segments is 2047 units.

Solution to 1.17  
(a) $\left( \frac{4}{5} \right)^{12} \ell_0$  
(b) $\frac{1 - \left( \frac{2}{5} \right)^{13}}{1 - \frac{4}{5}} \ell_0$  
(c) $5\ell_0$
Solution to 1.18

\( F_n = \frac{2}{r_0 \beta^n} \)

(b) \( \beta < 1 \)

Solution to 1.19

(a) 2097151 branches; volume 105.6 cm\(^3\); surface 9950.57 cm\(^2\)

(b) 0.625 < \( \beta < 0.7905 \)

Solution to 1.20

(a) \( A = 4.829 \text{ cm}^2 \)  

(b) \( A = 5.36 \text{ cm}^2 \)