Chapter 12
Taylor series

12.1 Introduction
The topic of this chapter is find approximations of functions in terms of power series, also called Taylor series. Such series can be described informally as infinite polynomials (i.e. polynomials containing infinitely many terms). Understanding when these objects are meaningful is also related to the issue of convergence, so our discussions will rely on the background assembled in the previous chapters.

The theme of approximation has appeared often in our calculus course. In a previous semester, we discussed a linear approximation to a function. The idea was to approximate the value of the function close to a point on its graph using a straight line (the tangent line). We noted in doing so that the approximation was good only close to the point of tangency. In general, further away, the graph of the functions curves away from that straight line. This leads naturally to the question: can we improve this approximation if we include other terms to describe this “curving away”? Here we extend such linear approximation methods. Our goal is to increase the accuracy of the linear approximation by including higher order terms (quadratic, cubic, etc), i.e. to find a polynomial that approximates the given function (see Figure 12.1). This idea forms an important goal in this chapter.

12.2 From geometric series to Taylor polynomials
In studying calculus, we explored a variety of functions. Among the most basic are polynomials, i.e. functions such as

\[ p(x) = x^5 + 2x^2 + 3x + 2. \]

Our introduction to differential calculus started with such functions for a reason: these functions are convenient and simple to handle. We found long ago that it is easy to compute derivatives of polynomials. The same can be said for integrals. One of our first examples, in Section 3.6.1 was the integral of a polynomial. We needed only use a power rule to integrate each term. An additional convenience of polynomials is that “evaluating” the function, (i.e. plugging in an \( x \) value and determining the corresponding \( y \) value) can be
done by simple multiplications and additions, i.e. by basic operations easily handled by an ordinary calculator. This is not the case for, say, trigonometric functions, exponential functions, or for that matter, most other functions we considered\(^ {61}\). For this reason, being able to approximate a function by a polynomial is an attractive proposition. This forms our main concern in the sections that follow.

We can arrive at connections between several functions and their polynomial approximations by exploiting our familiarity with the geometric series. We use both the results for convergence of the geometric series (from Chapter 11) and the formula for the sum of that series to derive a number of interesting results. Recall from Section 1.6.1 and Chapter 11 that the sum of an infinite geometric series is

\[
S = 1 + r + r^2 + \cdots + r^k + \cdots = \sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}, \quad \text{provided } |r| < 1. \quad (12.1)
\]

To connect this result to a statement about a function, we need a “variable”. Let us consider the behaviour of this series when we vary the quantity \(r\). To emphasize that \(r\) is now our variable, it will be helpful to change notation by substituting \(r = x\) into the above equation, while remembering that the formula in Equation (12.1) holds only provided that \(|r| = |x| < 1\).

\(^{61}\)For example, to find the decimal value of \(\sin(2.5)\) we would need a scientific calculator. These days the distinction is blurred, since powerful hand-held calculators are ubiquitous. Before such devices were available, the ease of evaluating polynomials made them even more important.
12.2.1 A simple expansion

Substitute the variable \( x = r \) into the series (12.1). Then formally, rewriting the above with this substitution, leads to the conclusion that

\[
\frac{1}{1-x} = 1 + x + x^2 + \ldots
\]  

(12.2)

We can think of this result as follows: Let

\[
f(x) = \frac{1}{1-x}
\]  

(12.3)

Then for \(-1 < x < 1\), it is true that \( f(x) \) can be approximated in terms of the polynomial

\[
p(x) = 1 + x + x^2 + \ldots
\]  

(12.4)

In other words, by Equation (12.1), the two expressions “are the same” for \(|x| < 1\), in the sense that the polynomial converges to the value of the function.

We refer to this \( p(x) \) as an (infinite) Taylor polynomial\(^{62}\) or simply a Taylor series for the function \( f(x) \). The usefulness of this kind of result can be illustrated by a simple example.

**Example:** Compute the value of the function \( f(x) = 1/(1-x) \) for \( x = 0.1 \) without using a calculator.

**Solution:** Plugging in the value \( x = 0.1 \) into the function directly leads to \( 1/(1-0.1) = 1/0.9 \), whose evaluation with no calculator requires long division.\(^{63}\) Using the polynomial representation, we have a simpler method:

\[
p(0.1) = 1 + 0.1 + 0.1^2 + \cdots = 1 + 0.1 + 0.01 + \cdots = 1.11\ldots
\]

\[\Diamond\]

We provide a few other examples based on substitutions of various sorts using the geometric series as a starting point.

12.2.2 A simple substitution

We make the substitution \( r = -t \), then \(|r| < 1\) means that \(|-t| = |t| < 1\), so that the formula (12.1) for the sum of a geometric series implies that:

\[
\frac{1}{1-(-t)} = 1 + (-t) + (-t)^2 + (-t)^3 + \ldots
\]

\[
\frac{1}{1+t} = 1 - t + t^2 - t^3 + t^4 + \ldots \quad \text{provided } |t| < 1
\]  

(12.5)

This means that we have produced a series expansion for the function \( 1/(1+t) \). We can go farther with this example by a new manipulation, whereby we integrate both sides to arrive at a new function and its expansion, shown next.

\(^{62}\)A Taylor polynomial contains finitely many terms, \( n \), whereas a Taylor series has \( n \to \infty \).

\(^{63}\)This example is slightly “trivial”, in the sense that evaluating the function itself is not very difficult. However, in other cases, we will find that the polynomial expansion is the only way to find the desired value.
12.2.3 An expansion for the logarithm

We use the result of Equation (12.5) and integrate both sides. On the left, we integrate the function \( f(t) = 1/(1 + t) \) (to arrive at a logarithm type integral) and on the right we integrate the power terms of the expansion. We are permitted to integrate the power series term by term provided that the series converges. This is an important restriction that we emphasize:

**Important:** Manipulation of infinite series by integration, differentiation, addition, multiplication, or any other “term by term” computation makes sense only so long as the original series converges.

Provided \(|t| < 1\), we have that

\[
\int_0^x \frac{1}{1 + t} \, dt = \int_0^x (1 - t + t^2 - t^3 + t^4 - \ldots) \, dt
\]

\[
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots
\]

This procedure has allowed us to find a series representation for a new function, \( \ln(1 + x) \).

\[
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}. \quad (12.6)
\]

The sigma notation appended on the right is a compact formula that represents the pattern of the terms. Recall that in Chapter 1, we have gotten thoroughly familiar with such summation notation.\(^{64}\)

**Example:** Evaluate the logarithm \( \ln(1 + x) \) for \( x = 0.25 \).

An expansion for the logarithm is definitely useful, in the sense that (without a scientific calculator or log-tables) it is not possible to easily calculate the value of this function at a given point. Of course, if you wanted to program a scientific calculator.

For \( x = 0.25 \), we cannot find \( \ln(1.25) \) using simple operations. However, the value of the first few terms of the series are computable by simple multiplication, division, and addition: \( 0.25 - 0.25^2/2 + 0.25^3/3 \approx 0.2239 \). A scientific calculator gives \( \ln(1.25) \approx 0.2231 \), so the approximation produced by only three terms of the series is already quite good. \( \diamond \)

**Example:** For which \( x \) is the series for \( \ln(1 + x) \) in Equation (12.6) expected to converge?

Retracing our steps from the beginning (see Section 12.2.2), we note that the value of \( t \) is not permitted to leave the interval \(|t| < 1\) so we also need \(|x| < 1\) in the integration step.\(^ {65}\) We certainly cannot expect the series for \( \ln(1 + x) \) to converge when \(|x| > 1\).

\(^{64}\)The summation notation is not crucial and should certainly not be memorized. We are usually interested only in the first few terms of such a series in any approximation of practical value. More important is the procedure that lead to the different series.

\(^{65}\)Strictly speaking, we should have ensured that we are inside this interval of convergence before we computed the last example.
Indeed, for $x = -1$, we have $\ln(1 + x) = \ln(0)$, which is undefined. Also note that for $x = -1$ the right hand side of (12.6) becomes

$$-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right).$$

This is, of course, the harmonic series (multiplied by -1). But we already know from Section 11.4.2 that the harmonic series diverges. Thus, we must avoid $x = -1$, since the expansion will not converge there, and neither is the function defined. This example illustrates that outside the interval of convergence, the series and the function become “meaningless”.

Example: An expansion for $\ln(2)$.

Strictly speaking, our analysis does not predict what happens if we substitute $x = 1$ into the expansion of the function found in Section 12.2.3, because this value of $x$ is outside of the permitted range $-1 < x < 1$ in which the Taylor series can be guaranteed to converge. It takes some deeper mathematics (Abel’s theorem) to prove that the result of this substitution actually makes sense, and converges, i.e. that

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

We state without proof here that the alternating harmonic series converges to $\ln(2)$.

Example: An expansion for $\arctan(x)$.

Suppose we make the substitution $r = -t^2$ into the geometric series formula, and recall that we need $|r| < 1$ for convergence. Then

$$\frac{1}{1 - (-t^2)} = 1 + (-t^2) + (-t^2)^2 + (-t^2)^3 + \ldots$$

$$\frac{1}{1 + t^2} = 1 - t^2 + t^4 - t^6 + t^8 + \ldots = \sum_{k=0}^{\infty} (-1)^n t^{2n}$$

This series converges provided $|t| < 1$. Now integrate both sides, and recall that the antiderivative of the function $1/(1 + t^2)$ is $\arctan(t)$. Then

$$\int_0^x \frac{1}{1 + t^2} \, dt = \int_0^x (1 - t^2 + t^4 - t^6 + t^8 + \ldots) \, dt$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)}. \quad (12.7)$$

Example: An expansion for $\pi$.

For a particular application, consider plugging in $x = 1$ into Equation (12.7). Then

$$\arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$
But \( \arctan(1) = \pi/4 \). Thus, we have found a way of computing the irrational number \( \pi \), namely

\[
\pi = 4 \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \right) = 4 \left( \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(2k-1)} \right).
\]

While it is true that this series converges, the convergence is slow. Check this by adding up the first 100 or 1000 terms of this series with a spreadsheet or other computer software. This means that it is not practical to use such a series as an approximation for \( \pi \). In particular, there are other series that converge to \( \pi \) very rapidly. Those are used in any practical application – including software packages, where it is not enough to store the first 16 or so digits.

### 12.3 Taylor Series: a systematic approach

In Section 12.2, we found a variety of Taylor series expansions directly from manipulations of the geometric series. Here we ask how such Taylor series can be constructed more systematically, if we are given a function that we want to approximate.

Suppose we have a function which we want to represent by a power series,

\[
f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{k=0}^{\infty} a_k x^k.
\]

Here we will use the function \( f(x) \) to directly determine the coefficients \( a_k \). To calculate \( a_0 \), let \( x = 0 \) and note that

\[
f(0) = a_0 + a_1 0 + a_2 0^2 + a_3 0^3 + \ldots = a_0.
\]

We conclude that

\[
a_0 = f(0).
\]

By differentiating both sides we find the following:

\[
f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + k a_k x^{k-1} + \ldots
\]

\[
f''(x) = 2a_2 + 2 \cdot 3 a_3 x + \ldots + (k-1) k a_k x^{k-2} + \ldots
\]

\[
f'''(x) = 2 \cdot 3 a_3 + \ldots + (k-2)(k-1) k a_k x^{k-3} + \ldots
\]

\[
f^{(k)}(x) = 1 \cdot 2 \cdot 3 \cdot 4 \ldots k a_k + \ldots
\]

Here we have used the notation \( f^{(k)}(x) \) to denote the \( k \)'th derivative of the function. Now evaluate each of the above derivatives at \( x = 0 \). Then

\[
f'(0) = a_1 \quad \Rightarrow \quad a_1 = f'(0)
\]

\[
f''(0) = 2a_2 \quad \Rightarrow \quad a_2 = \frac{f''(0)}{2}
\]

\[
f'''(0) = 2 \cdot 3 a_3 \quad \Rightarrow \quad a_3 = \frac{f'''(0)}{2 \cdot 3}
\]

\[
f^{(k)}(0) = k! a_k \quad \Rightarrow \quad a_k = \frac{f^{(k)}(0)}{k!}
\]

\[66\text{The development of this section was motivated by online notes by David Austin.}\]
This gives us a recipe for calculating all coefficients $a_k$. This means that if we can compute all the derivatives of the function $f(x)$, then we know the coefficients of the Taylor series as well:

\[ f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \]  

(12.8)

Because we have evaluated all the coefficients by the substitution $x = 0$, we say that the resulting power series is the Taylor series of the function centered at $x = 0$, which is sometimes also called the Maclaurin series.

### 12.3.1 Taylor series for the exponential function, $e^x$

Consider the function $f(x) = e^x$. All the derivatives of this function are equal to $e^x$. In particular,

\[ f^{(k)}(x) = e^x \implies f^{(k)}(0) = 1. \]

So that the coefficients of the Taylor series are

\[ a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!}. \]

Therefore the Taylor series for $e^x$ about $x = 0$ is

\[ 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots + \frac{x^k}{k!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \]

This is a very interesting series. We state here without proof that this series converges for all values of $x$. Further, the function defined by the series is in fact equal to $e^x$ that is,

\[ e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \sum_{k=0}^{\infty} \frac{x^k}{k!} \]  

(12.9)

The implication is that the function $e^x$ is completely determined (for all $x$ values) by its behaviour (i.e. derivatives of all orders) at $x = 0$. In other words, the value of the function at $x = 1,000,000$ is determined by the behaviour of the function around $x = 0$. This means that $e^x$ is a very special function with superior “predictable features”. If a function $f(x)$ agrees with its Taylor polynomial on a region $(-a, a)$, as was the case here, we say that $f$ is analytic on this region. It is known that $e^x$ is analytic for all $x$.

**Exercise:** We can use the results of Equation (12.9) to establish the fact that the exponential function grows “faster” than any power function $x^n$. That is the same as saying that the ratio of $e^x$ to $x^n$ (for any power $n$) increases with $x$. We leave this as an exercise for the reader.
We can also easily obtain a Taylor series expansion for functions related to $e^x$, without assembling the derivatives. We start with the result that

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \ldots = \sum_{k=0}^{\infty} \frac{u^k}{k!}$$

Then, for example, the substitution $u = x^2$ leads to

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{6} + \ldots = \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!}$$

### 12.3.2 Taylor series of $\sin x$

In this section we derive the Taylor series for $\sin x$. The function and its derivatives are

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad f^{(4)}(x) = \sin x, \ldots$$

After this, the cycle repeats, i.e. $f(x) = f^{(4)}(x), \quad f'(x) = f^{(5)}(x), \ldots$. This means that

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1, \ldots$$

and so on in a cyclic fashion. In other words,

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = -\frac{1}{3!}, \quad a_4 = 0, \quad a_5 = \frac{1}{5!}, \ldots$$

Thus,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$  

Note that all polynomials of the Taylor series of $\sin x$ contain only odd powers of $x$. This stems from the fact that $\sin x$ is an odd function, i.e. its graph is symmetric to rotation about the origin, a concept we discussed in an earlier term. We state here without proof that the function $\sin x$ is analytic, so that the expansion converges to the function for all $x$.

### 12.3.3 Taylor series of $\cos x$

The Taylor series for $\cos x$ could be found by a similar sequence of steps. But in this case, this is unnecessary: We already know the expansion for $\sin x$, so we can find the Taylor series for $\cos x$ by simple term by term differentiation. Since $\sin x$ is analytic, this is permitted for all $x$. We leave it as an exercise for the reader to show that

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$  

Since $\cos(x)$ has symmetry properties of an even function, we find that its Taylor series is composed of even powers of $x$ only.
12.4 Taylor polynomials as approximations

It is instructive to demonstrate how successive terms in a Taylor series expansion lead to approximations that improve (see Figure 12.2). Such approximations will be the topic of

Figure 12.2. An approximation of $\sin x$ by successive Taylor polynomials, $T_1, T_3, T_5, T_7$ of degree 1, 3, 5, 7, respectively. Taylor polynomials with a higher degree do a better job of approximating the function on a larger interval about $x = 0$.

the last computer laboratory exercise in this course.

Here we demonstrate this idea with the expansion for the function $\sin x$ that we just obtained. To see this, consider the sequence of polynomials $T_n(x)$

$$
T_1(x) = x,
$$

$$
T_3(x) = x - \frac{x^3}{3!},
$$

$$
T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!},
$$

$$
T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!},
$$

where the index $n$ indicates the order of the polynomial (highest power of $x$). These polynomials provide a better and better approximation to the function $\sin x$ close to $x = 0$. The first of these is just a linear (or tangent line) approximation that we had studied long ago. The second improves this with a quadratic approximation, etc. Figure 12.2 illustrates how the first few Taylor polynomials approximate the function $\sin x$ near $x = 0$. Observe that as we keep more terms, $n$, in the polynomial $T_n(x)$, the approximating curve “hugs” the
graph of $\sin x$ over a longer and longer range. The student will be asked to use the spreadsheet, together with some calculations as done in this section, to produce a composite graph similar to Figure 12.2 for other functions.

**Example:** Estimating the error of approximations. For a given value of $x$ close to the base point (at $x = 0$), the error in the approximation between the polynomials and the function is the vertical distance between the graphs of the polynomial and the function, here again choose $\sin x$ as an illustration. For example, at $x = 2$ radians $\sin(2) = 0.9093$ (as found on a scientific calculator). The approximations are:

(a) $T_1(2) = 2$, which is very inaccurate,
(b) $T_3(2) = 2 - 2^3/3! \approx 0.667$ which is too small,
(c) $T_5(2) \approx 0.9333$ that is much closer and
(d) $T_7(2) \approx 0.9079$ that is closer still.

In general, we can approximate the size of the error using the next term that would occur in the polynomial if we kept a higher order expansion. The details of estimating such errors is omitted from our discussion.

**12.4.1 Taylor series centered at $a$**

All Taylor series discussed so far were centered at $x = 0$. Now we want to extend the concept of Taylor series to derive a polynomial representation of a function $f(x)$ centered at an arbitrary point $x = a$. The crucial idea for the derivation of the Taylor series was that $f(x)$ and its derivatives are subsequently evaluated at $x = 0$ in order to obtain polynomials that represent better and better approximations of $f(x)$ in the vicinity of 0. In exactly the same spirit we can approximate $f(x)$ at an arbitrary point $x = a$:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k. \quad (12.10)$$

Alternatively, you can imagine to translate $f(x)$ horizontally by distance $a$ and apply the Taylor expansion centered at $y = 0$ for the translated function $f(y)$ with $y = x - a$.

**12.5 Applications of Taylor series**

In this section we illustrate some of the applications of Taylor series to problems that may be difficult to solve using other conventional methods. Some functions do not have an antiderivative that can be expressed in terms of other simple functions. Integrating these functions can be a problem, as we cannot use the Fundamental Theorem of Calculus. However, in some cases, we can approximate the value of the definite integral using a Taylor series. Another application of Taylor series is to compute an approximate solution to differential equations.
12.5. Applications of Taylor series

12.5.1 Evaluate an integral using Taylor series

Evaluate the definite integral
\[ \int_0^1 \sin(x^2) \, dx. \]

A simple substitution (e.g. \( u = x^2 \)) will not work here, and we cannot find an antiderivative. Here is how we might approach the problem using Taylor series: We know that the series expansion for \( \sin(t) \) is
\[ \sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots \]
Substituting \( t = x^2 \), we have
\[ \sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots \]
In spite of the fact that we cannot find an anti-derivative of the function, we can derive the anti-derivative of the Taylor series, term by term – just as we would for a long polynomial:
\[
\int_0^1 \sin(x^2) \, dx = \left. \left( x^3 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots \right) \right|_0^1 \\
= \left( \frac{x^3}{3} - \frac{x^6}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \ldots \right) \right|_0^1 \\
= \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \ldots 
\]
This is an alternating series with terms of decreasing magnitude and so we know that it converges. After adding up the first few terms, the pattern becomes clear and with some patience we find that the series converges to 0.310268.

12.5.2 Series solution of a differential equation

We are already familiar with the differential equation and initial condition that describes unlimited exponential growth.
\[
\frac{dy}{dx} = y, \\
y(0) = 1. 
\]
Indeed, from previous work, we know that the solution of this differential equation and initial condition is \( y(x) = e^x \), but we will pretend that we do not know this fact to illustrate the usefulness of Taylor series. In some cases, where separation of variables does not work, this option has great practical value.
Let us express the “unknown” solution to the differential equation as
\[ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots \]
Our task is to determine values for the coefficients $a_i$.

Since this function satisfies the condition $y(0) = 1$, we must have $y(0) = a_0 = 1$. Differentiating this power series leads to

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$$

But according to the differential equation, $\frac{dy}{dx} = y$. Hence, the two Taylor series must match, i.e.

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$$

In particular, this equality must hold for all values of $x$. However, this can only happen if the coefficients of like terms are the same, i.e. if the constant terms on either side of the equation are equal, if the terms of equal power of $x$ on either side are equal. Equating coefficients, we obtain:

$$a_0 = a_1 = 1, \quad \Rightarrow \quad a_1 = 1,$$

$$a_1 = 2a_2, \quad \Rightarrow \quad a_2 = \frac{a_1}{2} = \frac{1}{2},$$

$$a_2 = 3a_3, \quad \Rightarrow \quad a_3 = \frac{a_2}{3} = \frac{1}{2 \cdot 3},$$

$$a_3 = 4a_4, \quad \Rightarrow \quad a_4 = \frac{a_3}{4} = \frac{1}{2 \cdot 3 \cdot 4},$$

$$a_{n-1} = n a_n, \quad \Rightarrow \quad a_n = \frac{a_{n-1}}{n} = \frac{1}{1 \cdot 2 \cdot 3 \ldots n} = \frac{1}{n!}. \quad (12.11)$$

This means that

$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots = e^x,$$

which, as we have seen, is the expansion for the exponential function. This agrees with the solution we have been expecting. In the example here shown, we would hardly need to use series to arrive at the right conclusion. However, another example is discussed in the optional material, Section 12.7.1, where we would not find it as easy to discover the solution by other techniques discussed previously.

### 12.6 Summary

The main points of this chapter can be summarized as follows:

1. We discussed Taylor series and showed that some can be found using the formula for convergent geometric series. Two examples of Taylor series that were obtained in this way are

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad \text{for } |x| < 1$$

and

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \quad \text{for } |x| < 1$$
2. For Taylor series, we considered:
   (a) For what range of values of $x$ can we expect the series to converges?
   (b) Suppose we approximate the function on the right by a finite number of terms on the left. How good is that approximation?
   (c) If we include more and more such terms, does that approximation get better and better? (i.e., does the series converge to the function?)
   (d) Is the convergence rate rapid?

   Some of these questions occupy the attention of mathematicians, and are beyond the scope of an introductory calculus course.

3. More generally, we showed that the Taylor series for a function about $x = 0$,

   $$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{k=0}^{\infty} a_k x^k.$$  

   can be found by computing the coefficients

   $$a_k = \frac{f^{(k)}(0)}{k!}$$

4. As applications we discussed Taylor series to approximate a function, to find an approximation for a definite integral of a function, and to solve a differential equation.
12.7 Optional Material

12.7.1 Airy’s equation

Airy’s equation arises in the study of optics, and (with initial conditions) is as follows:

\[ y'' = xy, \quad y(0) = 1, \quad y'(0) = 0. \]

This differential equation cannot be easily solved using the integration techniques discussed in this course. However, we can solve it using Taylor series. As in Section 12.5.2, we write the solution as an arbitrary series:

\[ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots \]

Using the information from the initial conditions, we get \( y(0) = a_0 = 1 \) and \( y'(0) = a_1 = 0 \). Now we can write down the derivatives:

\[ y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots \]
\[ y'' = 2a_2 + 2 \cdot 3x + 3 \cdot 4x^2 + 4 \cdot 5x^3 + \ldots \]

The equation then gives

\[ y'' = xy \]

\[ 2a_2 + 2 \cdot 3a_3 x + 3 \cdot 4a_4 x^2 + 4 \cdot 5a_5 x^3 + \ldots = x(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots) \]
\[ 2a_2 + 2 \cdot 3a_3 x + 3 \cdot 4a_4 x^2 + 4 \cdot 5a_5 x^3 + \ldots = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \ldots \]

Again, we can equate the coefficients of \( x \), and use \( a_0 = 1 \) and \( a_1 = 0 \), to obtain

\[ 2a_2 = 0 \Rightarrow a_2 = 0, \]
\[ 2 \cdot 3a_3 = a_0 \quad \Rightarrow a_3 = \frac{1}{2 \cdot 3}, \]
\[ 3 \cdot 4a_4 = a_1 \Rightarrow a_4 = 0, \]
\[ 4 \cdot 5a_5 = a_2 \Rightarrow a_5 = 0, \]
\[ 5 \cdot 6a_6 = a_3 \Rightarrow a_6 = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6}. \]

This gives us the first few terms of the solution:

\[ y = 1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \ldots \]

If we continue in this way, we can determine any number of terms of the series.
12.8 Exercises

Exercise 12.1  The Taylor series for the function \( \sin(x) \) is given by
\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\]
Find the Taylor series of
\[
y = \cos(x)
\]
by differentiating this function. Using the spreadsheet, plot the following functions (on the same graph):
\[
y = \cos(x), \quad y = T_1(x), \quad y = T_2(x), \quad y = T_3(x), \quad y = T_4(x)
\]
where \( T_k \) is the polynomial made up of the first \( k \) (non-zero) terms in the Taylor series for \( \cos(x) \).

Exercise 12.2  An expansion for the function \( e^x \) is given by
\[
e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.
\]
Use the first seven terms of this series to estimate the value of the base of natural logarithms, i.e. of \( e^1 \).

Exercise 12.3  Find the Taylor series for each of the following functions about \( x=0 \):
(a) \( f(x) = x \cos(x) \),
(b) \( e^{-x^2} \)
(c) \( \frac{\sin(x)}{x} \)
(d) Use your results to find a Taylor series representation for the following integrals:
   (i) \( \int_{0}^{x} x \cos(x) \, dx \)
   (ii) \( \int_{0}^{x} e^{-x^2} \, dx \)

Exercise 12.4  Find the Taylor series at \( x=0 \) of the following functions:
(a) \( f(x) = e^{-x^2/2} \)
(b) \( f(x) = \ln(1 + x) \)
(c) \( f(x) = \sqrt{x} \sin \sqrt{x} \)
(d) \( f(x) = \frac{e^x + e^{-x}}{2} \)

Exercise 12.5  Using the Taylor series of \( f(x) \) near a point \( x^* \), the value of a smooth function \( f(x) \) in the vicinity of \( x^* \) can be approximated by \( f(x^*) \) and \( f^{(k)}(x^*) \) (where \( f^{(k)}(x) \) is the \( k \)th derivative of \( f(x) \)), and the distance between \( x \) and \( x^* \), \( \Delta x = x - x^* \):
\[
f(x) = f(x^*) + f'(x^*) \Delta x + \frac{f''(x^*)}{2!} (\Delta x)^2 + \frac{f'''(x^*)}{3!} (\Delta x)^3 + \cdots + \frac{f^{(k)}(x^*)}{k!} (\Delta x)^k + \cdots
\]
where \( k! = k \cdot (k - 1) \cdot (k - 2) \cdots 3 \cdot 2 \cdot 1 \) is called the factorial of the integer \( k \).

If \( |\Delta x| \ll 1 \) (i.e., \( x \) is very close to \( x^* \)), \( (\Delta x)^k \) becomes negligibly small. When keeping only the first \( k \) terms of the series, the error arising from throwing away all the terms subsequent to the \( k \)th term typically has a magnitude of the same order as \( |\Delta x|^{k+1} \). This error can be made arbitrarily small by keeping more terms in the series. Thus, Taylor series is often used in calculating the approximate value of a function that is not a polynomial.

For example, knowing that \( \sqrt{25} = 5 \), we can calculate an approximate value of \( \sqrt{26} \) by using the Taylor series. Note that,

\[
\sqrt{26} = \sqrt{25 + 1} = \sqrt{25(1 + \frac{1}{25})} = 5\sqrt{1 + \frac{1}{25}}.
\]

Now let \( x^* = 1 \) and \( x = 1 + \frac{1}{25} \), thus \( \Delta x = x - x^* = \frac{1}{25} = 0.04 \) which is small. Now using the Taylor series and keeping the first three terms:

\[
\sqrt{x} \approx \sqrt{x^*} + [(\sqrt{x})']_{x=x^*} \Delta x + \frac{[(\sqrt{x})']_{x=x^*} \Delta x^2}{2!}
\]

which yields

\[
\sqrt{1 + \frac{1}{25}} \approx 1 + \frac{1}{2} 0.04 - \frac{1}{8} 0.04^2 = 1.0198.
\]

Thus,

\[
\sqrt{26} = 5\sqrt{1 + \frac{1}{25}} \approx 5 \cdot 1.0198 = 5.099.
\]

Using a calculator, we find \( \sqrt{26} = 5.0990195 \ldots \), which is identical for the first 4 digits.

Use the first 3 terms of Taylor series to estimate the values of the following functions. Calculate by hand and compare the result with that obtained by a scientific calculator!

(a) \( \sqrt{110} \) \hspace{1cm} (Hint: \( 110 = 100 + 10 \) and \( \sqrt{100} = 10 \))
(b) \( e^{0.1} \) \hspace{1cm} (Hint: \( 0.1 = 0 + 0.1 \) and \( e^0 = 1 \))
(c) \( \cos(3) \) \hspace{1cm} (Hint: \( 3 = \pi - (\pi - 3) \approx \pi - 0.14 \) and \( \cos(\pi) = -1 \))
(d) \( \arctan(1.1) \) \hspace{1cm} (Hint: \( \arctan(1) = \pi/4, \ (\arctan(x))' = 1/(1 + x^2) \))

**Exercise 12.6**  Find the Taylor series at \( x = 0 \), up to and including the \( x^n \) term, for the following functions:

(a) \( f(x) = \tan x, \ n = 3 \) \hspace{1cm} (b) \( f(x) = (x + 1)e^x, \ n = 4 \)
(c) \( f(x) = e^{x^2-2x}, \ n = 3 \) \hspace{1cm} (d) \( f(x) = (\sin x)^2, \ n = 6 \)

**Exercise 12.7**

(a) Find a Taylor series for the function \( f(x) = 1/(1 + x^3) \) about \( x = 0 \). Show that this can be done by making the substitution \( r = -x^3 \) into the sum of a geometric series \( (S = \sum r^k = \frac{1}{1-r}) \).
(b) Use the same idea to find the Taylor series for the function \( f(t) = \frac{1}{1 + t^2} \).

(c) Use your result in part (b) to find a Taylor series for the function \( \arctan(x) \). (Hint: recall that \( \arctan(x) = \int_0^x \frac{1}{1 + t^2} \, dt \).

**Exercise 12.8** Let \( E(x) \) be the function defined by

\[
E(x) = \int_0^x \frac{e^{-t} - 1}{t} \, dt.
\]

(a) Use the Taylor series about \( t = 0 \) for the function \( e^{-t} \) to write down the analogous Taylor series about \( t = 0 \) for the function \( \frac{e^{-t} - 1}{t} \).

(b) Use your result in part (a) to determine the Taylor series about \( x = 0 \) for the integral \( E(x) \).

**Exercise 12.9** Growth of the exponential function Consider the exponential function \( y = f(x) = e^x \). Write down a Taylor Series series expansion for this function. Divide both sides by \( x^n \). Now examine the terms for the quantity \( e^x / x^n \). Use this expansion to argue that the exponential function grows faster than any power function.

**Exercise 12.10** Find a Taylor series about \( x = 0 \) for

\[
F(x) = \int_0^x \ln(1 + t^2) \, dt.
\]

**Exercise 12.11** Use a Taylor series representation to find the function \( y(t) \) that satisfies the differential equation \( y'(t) = y + bt \) with the initial condition \( y(0) = 1 \). This type of equation is called a non-homogeneous differential equation. Show that when \( b = 0 \) your answer agrees with the known exponential solution of the equation \( y'(t) = y \).

**Exercise 12.12** Use Taylor series to find the function that satisfies the following second order differential equation and initial conditions:

\[
\frac{d^2y}{dt^2} + y = 0, \quad y(0) = 0, \quad y'(0) = 1.
\]

A second order differential equation is one in which a second derivative appears. Notice that this type of differential equation comes with two initial conditions. Your answer should display \( y(t) \) as a Taylor series expansion of the desired function. (You may be able to then guess what elementary function has this expansion as its Taylor series.)
12.9 Solutions

Solution to 12.1 \( \cos(x) = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \) and hence

\[ T_1(x) = 1; \quad T_2(x) = 1 - \frac{x^2}{2!}; \quad T_3(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}; \quad T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}, \]

see Figure 12.3.

![Figure 12.3. Solution to problem 12.1.](image)

Solution to 12.2 \( e^1 \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \approx 2.7181. \)

Solution to 12.3

(a) \( x \cos(x) = x - \frac{x^3}{2} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} \).

(b) \( e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \).

(c) \( \frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} \).

(d.i) \( \int_0^x x \cos(x) \, dx = \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{6 \cdot 4!} - \frac{x^8}{8 \cdot 6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n+2}}{(2n+2)!} \).
(d.ii) \[ \int_0^x e^{-x^2} \, dx = x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2} - \frac{x^7}{7 \cdot 3!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} \]

**Solution to 12.4**

(a) \[ e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} \]

(b) \[ \ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \]

(c) \[ \sqrt{x} \sin(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(2n+1)!} \]

(d) \[ \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} \]

**Solution to 12.5**

(a) \( \sqrt{110} \approx 10.4875 \) (calculator: 10.488088).

(b) \( e^{0.1} = e^{0+0.1} \approx 1.105 \) (calculator: 1.1051709).

(c) \( \cos(3) = \cos(\pi + (-0.14)) \approx -0.9902 \) (calculator: -0.9899925).

(d) \( \arctan(1.1) = \arctan(1 + 0.1) \approx 0.83289816 \) (calculator: 0.83298127).

**Solution to 12.6**

(a) \( x + \frac{1}{3} x^3 \)

(b) \( 1 + 2x + \frac{3}{2} x^2 + \frac{2}{3} x^3 + \frac{5}{24} x^4 \)

(c) \( 1 - 2x + 3x^2 - \frac{10}{3} x^3 \)

(d) \( x^2 - \frac{x^4}{3} + \frac{2x^6}{45} \)

**Solution to 12.7**

(a) \( S = \frac{1}{1 + x^3} = \sum_{n=0}^{\infty} (-1)^n x^{3n} \)

(b) \( S = \frac{1}{1 + t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n} \)

(c) \( S = \arctan(x) = \int_0^x \frac{1}{1 + t^2} \, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \)

**Solution to 12.8**

(a) \( \frac{e^{-t} - 1}{t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} t^{n-1} \)

(b) \( E(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot n!} \)

**Solution to 12.9** \( \frac{e^x}{x^n} > \frac{x}{(n+1)!} \). For fixed \( n \), the lower bound becomes arbitrarily large for \( x \to \infty \) and hence \( e^x \) must grow faster than \( x^n \) for any \( n \).
Solution to 12.10 \( F(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n+1}}{n(2n+1)}. \)

Solution to 12.11 \( y(t) = 1 + (1 + b)t + \frac{1}{2}(1 + b)t^2 + \frac{1}{6}(1 + b)t^3 + \ldots \)
Setting \( b = 0 \) yields \( y(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots = e^t, \) as it should.

Solution to 12.12 \( y(t) = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots = \sin t. \)