Chapter 4

Applications of the definite integral to velocities and rates

4.1 Introduction

In this chapter, we encounter a number of applications of the definite integral to practical problems. We will discuss the connection between acceleration, velocity and displacement of a moving object, a topic we visited in an earlier, Differential Calculus Course. Here we will show that the notion of antiderivatives and integrals allows us to deduce details of the motion of an object from underlying Laws of Motion. We will consider both uniform and accelerated motion, and recall how air resistance can be described, and what effect it induces.

An important connection is made in this chapter between a rate of change (e.g. rate of growth) and the total change (i.e. the net change resulting from all the accumulation and loss over a time span). We show that such examples also involve the concept of integration, which, fundamentally, is a cumulative summation of infinitesimal changes. This allows us to extend the utility of the mathematical tools to a variety of novel situations. We will see examples of this type in Sections 4.3 and 4.4.

Several other important ideas are introduced in this chapter. We encounter for the first time the idea of spatial density, and see that integration can also be used to “add up” the total amount of material distributed over space. In Section 5.2.2, this idea is applied to the density of cars along a highway. We also consider mass distributions and the notion of a center of mass.

Finally, we also show that the definite integral is useful for determining the average value of a function, as discussed in Section 4.5. In all these examples, the important step is to properly set up the definite integral that corresponds to the desired net change. Computations at this stage are relatively straightforward to emphasize the process of setting up the appropriate integrals and understanding what they represent.
4.2 Displacement, velocity and acceleration

Recall from our study of derivatives that for $x(t)$ the position of some particle at time $t$, $v(t)$ its velocity, and $a(t)$ the acceleration, the following relationships hold:

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = a.$$  \hspace{1cm} (Velocity is the derivative of position and acceleration is the derivative of velocity.) This means that position is an anti-derivative of velocity and velocity is an anti-derivative of acceleration.

Since position, $x(t)$, is an anti-derivative of velocity, $v(t)$, by the Fundamental Theorem of Calculus, it follows that over the time interval $T_1 \leq t \leq T_2$,

$$\int_{T_1}^{T_2} v(t) \, dt = x(t) \bigg|_{T_1}^{T_2} = x(T_2) - x(T_1). \hspace{1cm} (4.1)$$

The quantity on the right hand side of Eqn. (4.1) is a displacement, i.e., the difference between the position at time $T_1$ and the position at time $T_2$. In the case that $T_1 = 0, T_2 = T$, we have

$$\int_0^{T} v(t) \, dt = x(T) - x(0),$$

as the displacement over the time interval $0 \leq t \leq T$.

Similarly, since velocity is an anti-derivative of acceleration, the Fundamental Theorem of Calculus says that

$$\int_{T_1}^{T_2} a(t) \, dt = v(t) \bigg|_{T_1}^{T_2} = v(T_2) - v(T_1). \hspace{1cm} (4.2)$$

as above, we also have that

$$\int_0^{T} a(t) \, dt = v(t) \bigg|_0^{T} = v(T) - v(0)$$

is the net change in velocity between time 0 and time $T$, (though this quantity does not have a special name).

4.2.1 Geometric interpretations

Suppose we are given a graph of the velocity $v(t)$, as shown on the left of Figure 4.1. Then by the definition of the definite integral, we can interpret $\int_{T_1}^{T_2} v(t) \, dt$ as the “area” associated with the curve (counting positive and negative contributions) between the endpoints $T_1$ and $T_2$. Then according to the above observations, this area represents the displacement of the particle between the two times $T_1$ and $T_2$.

Similarly, by previous remarks, the area under the curve $a(t)$ is a geometric quantity that represents the net change in the velocity, as shown on the right of Figure 4.1.

Next, we consider two examples where either the acceleration or the velocity is constant. We use the results above to compute the displacements in each case.
4.2. Displacement, velocity and acceleration

Figure 4.1. The total area under the velocity graph represents net displacement, and the total area under the graph of acceleration represents the net change in velocity over the interval $T_1 \leq t \leq T_2$.

### 4.2.2 Displacement for uniform motion

We first examine the simplest case that the velocity is constant, i.e. $v(t) = v = \text{constant}$. Then clearly, the acceleration is zero since $a = \frac{dv}{dt} = 0$ when $v$ is constant. Thus, by direct antidifferentiation,

$$\int_{0}^{T} v \, dt = vt\bigg|_{0}^{T} = v(T - 0) = vT.$$

However, applying result (4.1) over the time interval $0 \leq t \leq T$ also leads to

$$\int_{0}^{T} v \, dt = x(T) - x(0).$$

Therefore, it must be true that the two expressions obtained above must be equal, i.e.

$$x(T) - x(0) = vT.$$

Thus, for uniform motion, the displacement is proportional to the velocity and to the time elapsed. The final position is

$$x(T) = x(0) + vT.$$

This is true for all time $T$, so we can rewrite the results in terms of the more familiar (lower case) notation for time, $t$, i.e.

$$x(t) = x(0) + vt. \quad (4.3)$$

### 4.2.3 Uniformly accelerated motion

In this case, the acceleration $a$ is a constant. Thus, by direct antidifferentiation,

$$\int_{0}^{T} a \, dt = at\bigg|_{0}^{T} = a(T - 0) = aT.$$


However, using Equation (4.2) for $0 \leq t \leq T$ leads to

$$\int_0^T a \, dt = v(T) - v(0).$$

Since these two results must match, $v(T) - v(0) = aT$ so that

$$v(T) = v(0) + aT.$$

Let us refer to the initial velocity $V(0)$ as $v_0$. The above connection between velocity and acceleration holds for any final time $T$, i.e., it is true for all $t$ that:

$$v(t) = v_0 + at. \quad (4.4)$$

This just means that velocity at time $t$ is the initial velocity incremented by an increase (over the given time interval) due to the acceleration. From this we can find the displacement and position of the particle as follows: Let us call the initial position $x(0) = x_0$. Then

$$\int_0^T v(t) \, dt = x(T) - x_0. \quad (4.5)$$

But

$$I = \int_0^T v(t) \, dt = \int_0^T (v_0 + at) \, dt = \left. \left( v_0 t + \frac{a t^2}{2} \right) \right|_0^T = \left( v_0 T + \frac{a T^2}{2} \right). \quad (4.6)$$

So, setting Equations (4.5) and (4.6) equal means that

$$x(T) - x_0 = v_0 T + \frac{a T^2}{2}.$$

But this is true for all final times, $T$, i.e. this holds for any time $t$ so that

$$x(t) = x_0 + v_0 t + \frac{a t^2}{2}. \quad (4.7)$$

This expression represents the position of a particle at time $t$ given that it experienced a constant acceleration. The initial velocity $v_0$, initial position $x_0$ and acceleration $a$ allowed us to predict the position of the object $x(t)$ at any later time $t$. That is the meaning of Eqn. (4.7)\textsuperscript{17}.

### 4.2.4 Non-constant acceleration and terminal velocity

In general, the acceleration of a falling body is not actually uniform, because frictional forces impede that motion. A better approximation to the rate of change of velocity is given by the differential equation

$$\frac{dv}{dt} = g - kv. \quad (4.8)$$

\textsuperscript{17}Of course, Eqn. (4.7) only holds so long as the object is accelerating. Once the a falling object hits the ground, for example, this equation no longer holds.
4.2. Displacement, velocity and acceleration

We will assume that initially the velocity is zero, i.e. \( v(0) = 0 \).

This equation is a mathematical statement that relates changes in velocity \( v(t) \) to the constant acceleration due to gravity, \( g \), and drag forces due to friction with the atmosphere. A good approximation for such drag forces is the term \( kv \), proportional to the velocity, with \( k \), a positive constant, representing a frictional coefficient. Because \( v(t) \) appears both in the derivative and in the expression \( kv \), we cannot apply the methods developed in the previous section directly. That is, we do not have an expression that depends on time whose antiderivative we would calculate. The derivative of \( v(t) \) (on the left) is connected to the unknown \( v(t) \) on the right.

Finding the velocity and then the displacement for this type of motion requires special techniques. In Chapter 9, we will develop a systematic approach, called Separation of Variables to find analytic solutions to equations such as (4.8).

Here, we use a special procedure that allows us to determine the velocity in this case. We first recall the following result from first term calculus material:

The differential equation and initial condition

\[
\frac{dy}{dt} = -ky, \quad y(0) = y_0 \tag{4.9}
\]

has a solution

\[
y(t) = y_0 e^{-kt} \tag{4.10}
\]

Equation (4.8) implies that

\[ a(t) = g - kv(t), \]

where \( a(t) \) is the acceleration at time \( t \). Taking a derivative of both sides of this equation leads to

\[
\frac{da}{dt} = -k \frac{dv}{dt} = -ka.
\]

We observe that this equation has the same form as equation (4.9) (with \( a \) replacing \( y \)), which implies (according to 4.10) that \( a(t) \) is given by

\[
a(t) = C e^{-kt} = a_0 e^{-kt}.
\]

Initially, at time \( t = 0 \), the acceleration is \( a(0) = g \) (since \( a(t) = g - kv(t) \), and \( v(0) = 0 \). Therefore,

\[
a(t) = g e^{-kt}.
\]

Since we now have an explicit formula for acceleration vs time, we can apply direct integration as we did in the examples in Sections 4.2.2 and 4.2.3. The result is:

\[
\int_0^T a(t) \, dt = \int_0^T g e^{-kt} \, dt = g \int_0^T e^{-kt} \, dt = g \left[ \frac{e^{-kt}}{-k} \right]_0^T = g \left( \frac{e^{-kT} - 1}{-k} \right) = \frac{g}{k} (1 - e^{-kT}).
\]

In the calculation, we have used the fact that the antiderivative of \( e^{-kt} \) is \( e^{-kt} / k \). (This can be verified by simple differentiation.)
From our result here, we can also determine how the velocity behaves in the long term: observe that for \( t \to \infty \), the exponential term \( e^{-kt} \to 0 \), so that

\[
v(t) \to \frac{g}{k} (1 - \text{very small quantity}) \approx \frac{g}{k}.
\]

Thus, when drag forces are in effect, the falling object does not continue to accelerate indefinitely: it eventually attains a terminal velocity. We have seen that this limiting velocity is \( v = \frac{g}{k} \). The object continues to fall at this (approximately constant) speed as shown in Figure 4.2. The terminal velocity is also a steady state value of Eqn. (4.8), i.e., a value of the velocity at which no further change occurs.

### 4.3 From rates of change to total change

In this section, we examine several examples in which the rate of change of some process is specified. We use this information to obtain the total change\(^{18}\) that occurs over some time period.

\(^{18}\)We will use the terminology “total change” and “net change” interchangeably in this section.
4.3. From rates of change to total change

Changing temperature

We must carefully distinguish between information about the time dependence of some function, from information about the rate of change of some function. Here is an example of these two different cases, and how we would handle them.

(a) The temperature of a cup of juice is observed to be

\[ T(t) = 25(1 - e^{-0.1t}) \text{°Celcius} \]

where \( t \) is time in minutes. Find the change in the temperature of the juice between the times \( t = 1 \) and \( t = 5 \).

(b) The rate of change of temperature of a cup of coffee is observed to be

\[ f(t) = 8e^{-0.2t} \text{°Celcius per minute} \]

where \( t \) is time in minutes. What is the total change in the temperature between \( t = 1 \) and \( t = 5 \) minutes?

Solutions

(a) In this case, we are given the temperature as a function of time. To determine what net change occurred between times \( t = 1 \) and \( t = 5 \), we find the temperatures at each time point and subtract: That is, the change in temperature between times \( t = 1 \) and \( t = 5 \) is simply

\[ T(5) - T(1) = 25(1 - e^{-0.5}) - 25(1 - e^{-0.1}) = 25(0.94 - 0.606) = 7.47 \text{°Celcius}. \]

(b) Here, we do not know the temperature at any time, but we are given information about the rate of change. (Carefully note the subtle difference in the wording.) To get the total change, we would sum up all the small changes, \( f(t)\Delta t \) (over \( N \) subintervals of duration \( \Delta t = (5 - 1)/N = 4/N \)) for \( t \) starting at 1 and ending at 5 min. We obtain a sum of the form \( \sum f(t_k)\Delta t \) where \( t_k \) is the \( k \)’th time point. Finally, we take a limit as the number of subintervals increases (\( N \to \infty \)). By now, we recognize that this amounts to a process of integration. Based on this variation of the same concept we can take the usual shortcut of integrating the rate of change, \( f(t) \), from \( t = 1 \) to \( t = 5 \). To do so, we apply the Fundamental Theorem as before, reducing the amount of computation to finding antiderivatives. We compute:

\[
I = \int_{1}^{5} f(t) \, dt = \int_{1}^{5} 8e^{-0.2t} \, dt = -40e^{-0.2t} \bigg|_{1}^{5} = -40e^{-1} + 40e^{-0.2},
\]

\[
I = 40(e^{-0.2} - e^{-1}) = 40(0.8187 - 0.3678) = 18.
\]

Only in the second case did we need to use a definite integral to find a net change, since we were given the way that the rate of change depended on time. Recognizing the subtleties of the wording in such examples will be an important skill that the reader should gain.
4.3.1 Tree growth rates

The rate of growth in height for two species of trees (in feet per year) is shown in Figure 4.3. If the trees start at the same height, which tree is taller after 1 year? After 4 years?

Solution

In this problem we are provided with a sketch, rather than a formula for the growth rate of the trees. Our solution will thus be qualitative (i.e., descriptive), rather than quantitative. (This means we do not have to calculate anything; rather, we have to make some important observations about the behaviour shown in Fig 4.3.)

We recognize that the net change in height of each tree is of the form

\[ H_i(T) - H_i(0) = \int_0^T g_i(t) \, dt, \quad i = 1, 2. \]

where \( i = 1 \) for tree 1, \( i = 2 \) for tree 2, \( g_i(t) \) is the growth rate as a function of time (shown for each tree in Figure 4.3) and \( H_i(t) \) is the height of tree “i” at time \( t \). But, by the Fundamental Theorem of Calculus, this definite integral corresponds to the area under the curve \( g_i(t) \) from \( t = 0 \) to \( t = T \). Thus we must interpret the net change in height for each tree as the area under its growth curve. We see from Figure 4.3 that at \( t = 1 \) year, the area under the curve for tree 1 is greater, so it has grown more. At \( t = 4 \) years the area under the second curve is greatest so tree 2 has grown the most by that time.

4.3.2 Radius of a tree trunk

The trunk of a tree, assumed to have the shape of a cylinder, grows incrementally, so that its cross-section consists of “rings”. In years of plentiful rain and adequate nutrients, the tree grows faster than in years of drought or poor soil conditions. Suppose the rainfall pattern
4.3. From rates of change to total change

Figure 4.4. Rate of change of radius, \( f(t) \) for a growing tree over a period of 14 years.

has been cyclic, so that, over a period of 14 years, the growth rate of the radius of the tree trunk (in cm/year) is given by the function

\[
f(t) = 1.5 + \sin(\pi t/5),
\]

as shown in Figure 4.4. Let the height of the tree trunk be approximately constant over this ten year period, and assume that the density of the trunk is approximately 1 gm/cm\(^3\).

(a) If the radius was initially \( r_0 \) at time \( t = 0 \), what will the radius of the trunk be at time \( t \) later?

(b) What is the ratio of the mass of the tree trunk at \( t = 10 \) years and \( t = 0 \) years? (i.e. find the ratio mass(10)/mass(0).)

Solution

(a) Let \( R(t) \) denote the trunk’s radius at time \( t \). The rate of change of the radius of the tree is given by the function \( f(t) \), and we are told that at \( t = 0 \), \( R(0) = r_0 \). A graph of this growth rate over the first fifteen years is shown in Figure 4.4. The net change in the radius is

\[
R(t) - R(0) = \int_0^t f(s) \, ds = \int_0^t (1.5 + \sin(\pi s/5)) \, ds.
\]

Integrating the above, we get

\[
R(t) - R(0) = \left(1.5t - \frac{\cos(\pi s/5)}{\pi/5}\right)\bigg|_0^t.
\]

Here we have used the fact that the antiderivative of \( \sin(ax) \) is \(-\frac{(\cos(ax))/a}\).

Thus, using the initial value, \( R(0) = r_0 \) (which is a constant), and evaluating the integral, leads to

\[
R(t) = r_0 + 1.5t - \frac{5 \cos(\pi t/5)}{\pi} + \frac{5}{\pi}.
\]
(The constant at the end of the expression stems from the fact that $\cos(0) = 1$.) A graph of the radius of the tree over time (using $r_0 = 1$) is shown in Figure 4.5. This function is equivalent to the area associated with the function shown in Figure 4.4. Notice that Figure 4.5 confirms that the radius keeps growing over the entire period, but that its growth rate (slope of the curve) alternates between higher and lower values.

![Figure 4.5. The radius of the tree, $R(t)$, as a function of time, obtained by integrating the rate of change of radius shown in Fig. 4.4.](image)

After ten years we have

$$R(10) = r_0 + 15 - \frac{5}{\pi} \cos(2\pi) + \frac{5}{\pi}.$$  

But $\cos(2\pi) = 1$, so

$$R(10) = r_0 + 15.$$  

(b) The mass of the tree is density times volume, and since the density in this example is constant, 1 gm/cm$^3$, we need only obtain the volume at $t = 10$. Taking the trunk to be cylindrical means that the volume at any given time is

$$V(t) = \pi [R(t)]^2 h.$$  

The ratio we want is then

$$\frac{V(10)}{V(0)} = \frac{\pi [R(10)]^2 h}{\pi r_0^2 h} = \frac{[R(10)]^2}{r_0^2} = \left( \frac{r_0 + 15}{r_0} \right)^2.$$  

In this problem we used simple anti-differentiation to compute the desired total change. We also related the graph of the radial growth rate in Fig. 4.4 to that of the resulting radius at time $t$, in Fig. 4.5.
4.3.3 Birth rates and total births

After World War II, the birth rate in western countries increased dramatically. Suppose that the number of babies born (in millions per year) was given by

\[ b(t) = 5 + 2t, \quad 0 \leq t \leq 10, \]

where \( t \) is time in years after the end of the war.

(a) How many babies in total were born during this time period (i.e. in the first 10 years after the war)?

(b) Find the time \( T_0 \) such that the total number of babies born from the end of the war up to the time \( T_0 \) was precisely 14 million.

Solution

(a) To find the number of births, we would integrate the birth rate, \( b(t) \) over the given time period. The net change in the population due to births (neglecting deaths) is

\[ P(10) - P(0) = \int_0^{10} b(t) \, dt = \int_0^{10} (5 + 2t) \, dt = (5t + t^2)|_0^{10} = 50 + 100 = 150 \text{[million babies]}. \]

(b) Denote by \( T \) the time at which the total number of babies born was 14 million. Then, [in units of million]

\[ I = \int_0^T b(t) \, dt = 14 = \int_0^T (5 + 2t) \, dt = 5T + T^2 \]

equating \( I = 14 \) leads to the quadratic equation, \( T^2 + 5T - 14 = 0 \), which can be written in the factored form, \( (T - 2)(T + 7) = 0 \). This has two solutions, but we reject \( T = -7 \) since we are looking for time after the War. Thus we find that \( T = 2 \) years, i.e. it took two years for 14 million babies to have been born.

While this problem involves simple integration, we had to solve for a quantity (\( T \)) based on information about behaviour of that integral. Many problems in real application involve such slight twists on the ideas of integration.

4.4 Production and removal

The process of integration can be used to convert rates of production and removal into net amounts present at a given time. The example in this section is of this type. We investigate a process in which a substance accumulates as it is being produced, but disappears through some removal process. We would like to determine when the quantity of material increases, and when it decreases.
Circadean rhythm in hormone levels

Consider a hormone whose level in the blood at time \( t \) will be denoted by \( H(t) \). We will assume that the level of hormone is regulated by two separate processes: one might be the secretion rate of specialized cells that produce the hormone. (The production rate of hormone might depend on the time of day, in some cyclic pattern that repeats every 24 hours or so.) This type of cyclic pattern is called circadean rhythm. A competing process might be the removal of hormone (or its deactivation by some enzymes secreted by other cells.) In this example, we will assume that both the production rate, \( p(t) \), and the removal rate, \( r(t) \), of the hormone are time-dependent, periodic functions with somewhat different phases.

![Hormone production/removal rates graph](image)

**Figure 4.6.** The rate of hormone production \( p(t) \) and the rate of removal \( r(t) \) are shown here. We want to use these graphs to deduce when the level of hormone is highest and lowest.

A typical example of two such functions are shown in Figure 4.6. This figure shows the production and removal rates over a period of 24 hours, starting at midnight. Our first task will be to use properties of the graph in Figure 4.6 to answer the following questions:

1. When is the production rate, \( p(t) \), maximal?
2. When is the removal rate \( r(t) \) minimal?
3. At what time is the hormone level in the blood highest?
4. At what time is the hormone level in the blood lowest?
5. Find the maximal level of hormone in the blood over the period shown, assuming that its basal (lowest) level is \( H = 0 \).

**Solutions**

1. We see directly from Fig. 4.6 that production rate is maximal at about 9:00 am.
4.4. Production and removal

2. Similarly, removal rate is minimal at noon.

3. To answer this question we note that the total amount of hormone produced over a time period \( a \leq t \leq b \) is

\[ P_{\text{total}} = \int_{a}^{b} p(t) \, dt. \]

The total amount removed over time interval \( a \leq t \leq b \) is

\[ R_{\text{total}} = \int_{a}^{b} r(t) \, dt. \]

This means that the net change in hormone level over the given time interval (amount produced minus amount removed) is

\[ H(b) - H(a) = P_{\text{total}} - R_{\text{total}} = \int_{a}^{b} (p(t) - r(t)) \, dt. \]

We interpret this integral as the area between the curves \( p(t) \) and \( r(t) \). But we must use caution here: For any time interval over which \( p(t) > r(t) \), this integral will be positive, and the hormone level will have increased. Otherwise, if \( r(t) < p(t) \), the integral yields a negative result, so that the hormone level has decreased. This makes simple intuitive sense: If production is greater than removal, the level of the substance is accumulating, whereas in the opposite situation, the substance is decreasing. With these remarks, we find that the hormone level in the blood will be greatest at 3:00 pm, when the greatest (positive) area between the two curves has been obtained.

4. Similarly, the least hormone level occurs after a period in which the removal rate has been larger than production for the longest stretch. This occurs at 3:00 am, just as the curves cross each other.

5. Here we will practice integration by actually fitting some cyclic functions to the graphs shown in Figure 4.6. Our first observation is that if the length of the cycle (also called the period) is 24 hours, then the frequency of the oscillation is \( \omega = \frac{2\pi}{24} = \frac{\pi}{12} \). We further observe that the functions shown in the Figure 4.7 have the form

\[ p(t) = A(1 + \sin(\omega t)), \quad r(t) = A(1 + \cos(\omega t)). \]

Intersection points occur when

\[ p(t) = r(t) \]

\[ A(1 + \sin(\omega t)) = A(1 + \cos(\omega t)) \]

\[ \sin(\omega t) = \cos(\omega t) \]

\[ \tan(\omega t) = 1. \]

This last equality leads to \( \omega t = \frac{\pi}{4}, \frac{5\pi}{4} \). But then, the fact that \( \omega = \frac{\pi}{12} \) implies that \( t = 3, 15 \). Thus, over the time period \( 3 \leq t \leq 15 \) hrs, the hormone level is
increasing. For simplicity, we will take the amplitude $A = 1$. (In general, this would just be a multiplicative constant in whatever solution we compute.) Then the net increase in hormone over this period is calculated from the integral

$$H_{\text{total}} = \int_{3}^{15} [p(t) - r(t)] \, dt = \int_{3}^{15} [(1 + \sin(\omega t)) - (1 + \cos(\omega t))] \, dt$$

In the problem set, the reader is asked to compute this integral and to show that the amount of hormone that accumulated over the time interval $3 \leq t \leq 15$, i.e. between 3:00 am and 3:00 pm is $24\sqrt{2}/\pi$.

### 4.5 Average value of a function

In this final example, we apply the definite integral to computing the average height of a function over some interval. First, we define what is meant by average value in this context.\(^{19}\)

Given a function

$$y = f(x)$$

over some interval $a \leq x \leq b$, we will define average value of the function as follows:

**Definition**

The average value of $f(x)$ over the interval $a \leq x \leq b$ is

$$\bar{f} = \frac{1}{b - a} \int_{a}^{b} f(x) \, dx. \quad (4.12)$$

\(^{19}\)In Chapters 5 and 8, we will encounter a different type of average (also called mean) that will represent an average horizontal position or center of mass. It is important to avoid confusing these distinct notions.
Example 1

Find the average value of the function \( y = f(x) = x^2 \) over the interval \( 2 < x < 4 \).

Solution

\[
\bar{f} = \frac{1}{4 - 2} \int_2^4 x^2 \, dx = \frac{1}{2} \left[ \frac{x^3}{3} \right]_2^4 = \frac{1}{6} (4^3 - 2^3) = \frac{28}{3}
\]

Example 2: Day length over the year

Suppose we want to know the average length of the day during summer and spring. We will assume that day length follows a simple periodic behaviour, with a cycle length of 1 year (365 days). Let us measure time \( t \) in days, with \( t = 0 \) at the spring equinox, i.e. the date in spring when night and day lengths are equal, each being 12 hrs. We will refer to the number of daylight hours on day \( t \) by the function \( f(t) \). (We will also call \( f(t) \) the day-length on day \( t \).)

A simple function that would describe the cyclic changes of day length over the seasons is

\[
f(t) = 12 + 4 \sin \left( \frac{2\pi t}{365} \right).
\]

This is a periodic function with period 365 days as shown in Figure 4.8. Its maximal value is 16h and its minimal value is 8h. The average day length over spring and summer, i.e. over the first (365/2) \( \approx 182 \) days is:
\[ \bar{f} = \frac{1}{182} \int_{0}^{182} f(t) \, dt \]

\[ = \frac{1}{182} \int_{0}^{182} \left( 12 + 4 \sin \left( \frac{\pi t}{182} \right) \right) \, dt \]

\[ = \frac{1}{182} \left[ 12t - 4 \cdot \frac{182}{\pi} \cos \left( \frac{\pi t}{182} \right) \right]_{0}^{182} \]

\[ = \frac{1}{182} \left( 12 \cdot 182 - 4 \cdot \frac{182}{\pi} \left[ \cos(\pi) - \cos(0) \right] \right) \]

\[ = 12 + \frac{8}{\pi} \approx 14.546 \text{ hours} \quad (4.13) \]

Thus, on average, the day is 14.546 hrs long during the spring and summer.

In Figure 4.8, we show the entire day length cycle over one year. It is left as an exercise for the reader to show that the average value of \( f \) over the entire year is 12 hrs. (This makes intuitive sense, since overall, the short days in winter will average out with the longer days in summer.)

Figure 4.8 also shows geometrically what the average value of the function represents. Suppose we determine the area associated with the graph of \( f(x) \) over the interval of interest. (This area is painted red (dark) in Figure 4.8, where the interval is \( 0 \leq t \leq 365 \), i.e. the whole year.) Now let us draw in a rectangle over the same interval \((0 \leq t \leq 365)\) having the same total area. (See the big rectangle in Figure 4.8, and note that its area matches with the darker, red region.) The height of the rectangle represents the average value of \( f(t) \) over the interval of interest.

### 4.6 Application: Flu Vaccination

Suppose Health Canada has been monitoring flu outbreaks continuously over the last 100 years. They have found that the number of infections follows an annual (seasonal) cycle and a twenty-year-cycle. In all, the number of infections \( I(t) \) are well-approximated by the function:

\[ I(t) = \cos \left( \frac{\pi}{6} t \right) + \cos \left( \frac{\pi}{120} t \right) + 2 \quad (4.14) \]

where \( t \) is measured in months and \( I(t) \) given in units of 100,000 individuals. Figure 4.9 depicts the number of infections \( I(t) \) over time and illustrates the superposition of the annual cycle of seasonal flu outbreaks modulated by slower fluctuations with a longer period of 10 years.

Health Canada decides to eradicate the flu. This is estimated to take 5 years of intensive vaccination and quarantine. In order to use the least amount of resources, they decide to start their eradication program at the start of a 5-year term where the flu virus is naturally at a 5-year minimum average. Supposing that \( t = 0 \) is January 1\(^{st}\), 2013, when should they start this program?
Figure 4.9. Flu infections over time – the number of infections undergoes seasonal fluctuations, which are superimposed on slower fluctuations of a twenty-year-cycle (dashed line).

Solution

**Average:** First, find 5-year average, \( \bar{I}(t) \), starting at \( t \):

\[
\bar{I}(t) = \frac{1}{60} \int_t^{t+60} I(s) \, ds = \frac{1}{60} \int_t^{t+60} \cos \left( \frac{\pi}{6} t \right) + \cos \left( \frac{\pi}{120} t \right) + 2 \, ds
\]

\[
= \frac{1}{60} \left[ \sin \left( \frac{\pi}{6} s \right) \frac{6}{\pi} + \sin \left( \frac{\pi}{120} s \right) \frac{120}{\pi} + 2s \right]_{t}^{t+60}
\]

\[
= \frac{1}{60} \left[ \sin \left( \frac{\pi}{6} (t+60) \right) \frac{6}{\pi} + \sin \left( \frac{\pi}{120} (t+60) \right) \frac{120}{\pi} + 120 \right.
\]

\[
- \sin \left( \frac{\pi}{6} t \right) \frac{6}{\pi} - \sin \left( \frac{\pi}{120} t \right) \frac{120}{\pi} \right].
\]

Equation (4.15) could be further simplified using trigonometric identities but for our purposes this will do. \( \bar{I}(t) \) is shown in Figure 4.10.

**Minimum:** Second, find the minimum of the 5-year average by solving \( \frac{d\bar{I}(t)}{dt} = 0 \):

\[
\frac{d\bar{I}(t)}{dt} = \frac{1}{60} \left[ \cos \left( \frac{\pi}{6} (t+60) \right) + \cos \left( \frac{\pi}{120} (t+60) \right) - \cos \left( \frac{\pi}{6} t \right) - \cos \left( \frac{\pi}{120} t \right) \right]
\]

\[
= \frac{1}{60} \left[ \cos \left( \frac{\pi}{6} t \right) \cos(10\pi) - \sin \left( \frac{\pi}{6} t \right) \sin(10\pi) + \cos \left( \frac{\pi}{120} t \right) \cos \left( \frac{\pi}{2} \right) \right.
\]

\[
- \sin \left( \frac{\pi}{120} t \right) \sin \left( \frac{\pi}{2} \right) - \cos \left( \frac{\pi}{6} t \right) - \cos \left( \frac{\pi}{120} t \right) \left. \right]
\]

\[
= \frac{1}{60} \left[ - \sin \left( \frac{\pi}{120} t \right) - \cos \left( \frac{\pi}{120} t \right) \right].
\]

(4.16)
Figure 4.10. 5-year average of flu infections over time (red line), averaging starts at $t$ until $t + 60$ months. As a reference, the number of infections are also shown (black line).

Note, for the second equality we have used the trigonometric identity $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$. Hence, the start of a 5-year minimum (or maximum) average period is marked by the condition $\cos \left( \frac{\pi}{120} t \right) = -\sin \left( \frac{\pi}{120} t \right)$. The equality $\cos \alpha = -\sin \alpha$ holds for $\alpha = \frac{3\pi}{4}$ and $\alpha = \frac{7\pi}{4}$ (as well as when adding multiples of $2\pi$ to $\alpha$). Since $\cos(\alpha)$ is a decreasing function for $0 < \alpha < \pi$, we expect that $\alpha = \frac{3\pi}{4}$ marks a minimum. Solving

$$\frac{\pi}{120} t = \frac{3\pi}{4}$$

yields $t = 90$ months. Indeed this indicates the start of a 5-year minimum average because

$$\left. \frac{d^2 \bar{I}(t)}{dt^2} \right|_{t=90} = \frac{\pi}{120} > 0.$$ 

Hence the earliest intervention could start on June 1st, 2020, i.e. 90 months after January 1st, 2013.

Shortcut: Even though there is nothing wrong with the previous two steps, it is nevertheless important and educational to realize that we first calculated an antiderivative only to take the derivative of it! While this is a good exercise it is actually unnecessary and we could have used the Fundamental Theorem of Calculus (FTC) instead:

$$\frac{dI(t)}{dt} = \frac{d}{dt} \left[ \frac{1}{60} \int_{t}^{t+60} I(s) \, ds \right]$$

$$= \frac{1}{60} \frac{d}{dt} \left[ F(t + 60) - F(t) \right], \quad \text{where } F(t) \text{ is any antiderivative of } I(t)$$

$$= \frac{1}{60} \left[ I(t + 60) - I(t) \right].$$  \hspace{1cm} (4.17)
Note, for the second equality we have used the FTC part II and for the last equality the FTC, part I. After inserting \( I(t) \) and some algebraic manipulations this immediately leads to Equation (4.16) and the remaining calculations remain the same as before.

**Check:** Looking at Figure 4.9 we note that the the slow fluctuations (dashed line) have a minimum after 10 years. Therefore we would expect that the 5-year minimum average would be centered at 10 years and hence that the averaging window starts 2.5 years earlier, i.e. after 7.5 years or 90 months. In the present case this estimate turns out to be accurate – however, this is only true because in the present case the peak of the slow fluctuations coincides with the peak of seasonal flu infections.

## 4.7 Summary

In this chapter, we arrived at a number of practical applications of the definite integral.

1. In Section 4.2, we found that for motion at constant acceleration \( a \), the displacement of a moving object can be obtained by integrating twice: the definite integral of acceleration is the velocity \( v(t) \), and the definite integral of the velocity is the displacement.

\[
v(t) = v_0 + \int_0^t a \, ds. \quad x(t) = x(0) + \int_0^t v(s) \, ds.
\]

(Here we use the “dummy variable” \( s \) inside the integral, but the meaning is, of course, the same as in the previous presentation of the formulae.) We showed that at any time \( t \), the position of an object initially at \( x_0 \) with velocity \( v_0 \) is

\[
x(t) = x_0 + v_0 t + \frac{a t^2}{2}.
\]

2. We extended our analysis of a moving object to the case of non-constant acceleration (Section 4.2.4), when air resistance tends to produce a drag force to slow the motion of a falling object. We found that in that case, the acceleration gradually decreases, \( a(t) = ge^{-kt} \). (The decaying exponential means that \( a \to 0 \) as \( t \) increases.) Again, using the definite integral, we could compute a velocity,

\[
v(t) = \int_0^t a(s) \, ds = \frac{g}{k} (1 - e^{-kt}).
\]

3. We illustrated the connection between rates of change (over time) and total change (between on time point and another). In general, we saw that if \( r(t) \) represents a rate of change of some process, then

\[
\int_a^b r(s) \, ds = \text{Total change over the time interval } a \leq t \leq b.
\]

This idea was discussed in Section 4.3.
4. In the concluding Section 4.5, we discussed the average value of a function $f(x)$ over some interval $a \leq x \leq b$,

$$
\bar{f} = \frac{1}{b - a} \int_{a}^{b} f(x) \, dx.
$$

In the next few chapters we encounters a vast assortment of further examples and practical applications of the definite integral, to such topics as mass, volumes, length, etc. In some of these we will be called on to “dissect” a geometric shape into pieces that are not simple rectangles. The essential idea of an integral as a sum of many (infinitesimally) small pieces will, nevertheless be the same.
4.8 Exercises

Exercise 4.1  Two cars, labeled 1 and 2 start side by side and accelerate from rest. Figure 4.11 shows a graph of their velocity functions, with \( t \) measured in minutes.

![Figure 4.11. For problem 4.1](image)

(a) At what time(s) do the cars have equal velocities?
(b) At what time(s) do the cars have equal accelerations?
(c) Which car is ahead after one minute?
(d) Which car is ahead after 3 minutes?
(e) When does one car overtake the other? (Give an approximate answer)

Exercise 4.2  The speed of a car (km/h) is given by the expression

\[ v(t) = 2t^2 + 5t, \quad 0 < t < 1 \]

where \( t \) is time in hours. Use this expression to find

(a) The acceleration over this time period.
(b) The total displacement over the same time period.

Exercise 4.3  The velocity of a boat moving through water is found to be \( v(t) = 10(1 - e^{-t}) \).

(a) Find the acceleration of the boat and show that it satisfies a differential equation, i.e an equation of the form \( \frac{d[a(t)]}{dt} = -k \cdot a(t) \) for some value of the constant \( k \) (i.e. find that value of \( k \)).
(b) Find the displacement of the boat at time \( t \).
Exercise 4.4  A particle in a force field accelerates so that its acceleration is described by the function
\[ a(t) = t - t^2, \quad 0 < t < 1 \]
(a) Find the velocity of the particle \( v(t) \) for \( 0 < t < 1 \) assuming that it starts at rest.
(b) Find the total displacement of the particle over the time interval \( 0 < t < 1 \).

Exercise 4.5  An express mail truck delivers mail to various companies situated along a central avenue and often goes back and forth as new mail arrives. Over some period of time, \( 0 < t < 10 \), its velocity (in km per hour) can be described by the function:
\[ v(t) = t^2 - 9t + 14. \]
(a) Find the displacement over this period of time. (Hint: recall that if you leave home in the morning, travel to work, and then go back home, then your net distance traveled, or total displacement, over this full period of time is zero.)
(b) How much gasoline was consumed during this period of time if the vehicle uses \( 1/2 \) liters per km. (Hint: To answer (b), you will need to find the total distance that the vehicle actually covered during its trip.)

Exercise 4.6  The reaction time of a driver (time it takes to notice and react to danger in the road ahead) is about 0.5 seconds. When the brakes are applied, it then takes the car some time to decelerate and come to a full stop. If the deceleration rate is \( a = -8 \) m/sec\(^2\), how long would it take the driver to stop from an initial speed of 100 km per hour? (Include both reaction time and deceleration time, and use the Fundamental Theorem of Calculus to arrive at your answer.)

Exercise 4.7  You are driving your car quickly (at speed 100 km/h) to catch your flight to Hawaii for mid-term break. A pedestrian runs across the road, forcing you to brake hard. Suppose it takes you 1 second to react to the danger, and that when you apply your brakes, you slow down at the rate \( a = -10 \) m/s\(^2\).
(a) How long will it take you to stop?
(b) How far will your car move from the instant that the danger is sighted until coming to a complete stop?
Use the Fundamental Theorem of Calculus to arrive at your answer.

Exercise 4.8  The growth rate of a pair of twins (in mm/day) is shown in Figure 4.12. Suppose both children have the same weight at time \( t = 0 \).
(a) Which one is taller at age 0.5 years?
(b) Which one is taller at age 1 year?
Exercise 4.9  The flow rate of blood through the heart can be described approximately as a periodic function of the form \( F(t) = A(1 + \sin(0.15t)) \), where \( t \) is time in seconds and \( A \) is a constant in units of cubic cm per second. (Thus, at time \( t \), \( F(t) \) cm\(^3\) of blood flow through the heart per second.) Find the total volume of blood that flows through the heart between \( t = 0 \) and \( t = 1 \). (Express your answer in terms of \( A \).)

Exercise 4.10  During a gambling session lasting 6 hours, the rate of winnings at a casino in dollars per hour are seen to follow the formula \( w(t) = 2000t(1 - (t/6)) \). Find the total winnings during that whole session.

Exercise 4.11  At time \( t \), an intravenous infusion delivers a flow rate of \( y = 100(1 - t^3) \text{ cm}^3/\text{hr} \), where \( t \) is time in hours. The infusion contains a drug at concentration 0.1 mg/cm\(^3\). Find the total volume of fluid and the total amount of drug delivered to the patient over a 1 hour period from \( t = 0 \) to \( t = 1 \).

Exercise 4.12  Oil leaks out of an oil tanker at the rate \( f(t) = 10 - 0.2t^2 \) (where \( f \) is in 10 thousand barrels per hour and \( t \) is in hours). (Note: This function only makes sense as long as \( f(t) \geq 0 \) since a negative flow of oil is meaningless in this case.)

(a) At what time will the flow be zero?

(b) What is the total amount that has leaked out between \( t = 0 \) and the time you found in (a)?

Exercise 4.13  After a heavy rainfall, the rate of flow of water into a lake is found to satisfy the relationship \( F(t) = 4 - \left( \frac{t}{10} - 1 \right)^2 \) where \( t \) is time in hours, \( 0 \leq t \leq 30 \) and \( F \) is in units of 100,000 gallons/h.

(a) Find the time, \( t_1 \) at which the rate of flow is greatest.
(b) Find the time, $t_2$ at which the flow is zero.

(c) Find how much water in total has flowed into the lake between these two times.

**Exercise 4.14** The growth rate of a crop is known to depend on temperature during the growing season. Suppose the growth rate of the crop in tons per day is given by $g(t) = 0.1(T(t) - 18)$ where $T(t)$ is temperature in degrees Celsius. Suppose the temperature record during the 90 days of the season was $T(t) = 22 + 0.1t + 4\sin(2\pi t/60)$ where $t$ is time in days. Find the total growth (in tons) that would have occurred over the whole season.

**Exercise 4.15** This question concerns cumulative exposure to radiation experienced by people living near nuclear waste disposal sites.

(a) Recall from last term that radioactive material decays according to a negative exponential: $m(t) = m_0 \cdot e^{-rt}$, where $m(t)$ is the mass of the radioactive material at time $t$, $m_0 = m(0)$ is the initial mass at time $t = 0$, and $r$ is the rate of decay. The half-life is the time it takes for the material to decay to one half its initial mass. Suppose that $t$ is measured in months. Determine the rate of decay $r$ if the half-life is 1 month.

(b) Assume that at any time, the amount of radiation is proportional to the mass of the radioactive material. If initially the radiation level is 0.5 rads per month, how could we describe the radiation level as a function of time?

(c) Assume now that there is radioactive material in your backyard of the type considered in (a) above, i.e. that it has a half-life of one month. Calculate the cumulative exposure in rads that would occur if you lived in your house for 10 years.

**Exercise 4.16** The level of glucose in the blood depends on the rate of intake from ingestion of food and on the rate of clearance due to glucose metabolism. Shown in Figure 4.13 are two functions, $I(t)$ and $C(t)$ for the intake and clearance rates over a period of time after fasting. Both rates are functions of time $t$. Suppose that at time 0 there is no glucose in the blood.

(a) Express the level of blood glucose as a definite integral.

(b) At what approximate time was the intake rate maximal?

(c) At what approximate time was the clearance rate maximal?

(d) When was the blood level of glucose maximal?

**Exercise 4.17** The rate at which water flows in and out of Capilano Reservoir is described by two functions. $I(t)$ is the rate at which water flows in to the reservoir (in gallons per day) and $O(t)$ is the rate at which water flows out (in gallons per day). See sketch below in Figure 4.14. Assume that there is water in the reservoir at time $t = 0$.

(a) Express the quantity of water $Q(t)$ in the reservoir as a definite integral. (i.e. $Q(0) > 0$).
Exercise 4.18  The rate at which animals migrate into and out of a wildlife reserve is described by two functions shown in Figure 4.15. \( I(t) \) is the rate at which animals enter the reserve and \( O(t) \) is the rate at which they leave (both in number per day).

(a) Express the number of animals in the reserve as a definite integral.

(b) When is the number of animals in the reserve greatest and when is it smallest?

Exercise 4.19  During a particularly soggy week in Vancouver, rainfall reached epic proportions. The rainfall pattern was as follows: A constant 20 mm over the first day, a steady increase from 20 up to 50 mm over the next day (assume linear increase), 50 mm rain over the next day, a steady drop from 50 down to 40 mm over the next day, and a flat 30 mm over the next day.

(a) Determine the total amount of rain during this period.
Exercise 4.20  [98 Final] Find the average value of the function \( f(x) = \sin\left(\frac{\pi x}{2}\right) \) over the interval \([0, 2]\).

Exercise 4.21

(a) Find the average value of \( x^n \) over the interval \([0, 1]\).
(b) What happens as \( n \) becomes arbitrarily large (that is, \( n \to \infty \))? 
(c) Explain your answer to part (b) by considering the graphs of these functions.
(d) Repeat parts (a) - (c) using the functions \( x^{1/n} \).

Exercise 4.22  An object starts from rest at \( t = 0 \) and accelerates so that \( a = \frac{dv}{dt} = ce^{-t} \) where \( c \) is a positive constant.

(a) Find the average velocity over the first \( t \) seconds.
(b) What happens to this average velocity as \( t \) becomes very large?

Exercise 4.23  Symmetry

(a) Find the average value of the function \( \sin x \) over the interval \([-\pi, \pi]\).
(b) Find the average value of the function \( x^3 \) over the interval \([-1, 1]\).
(c) Find the average value of the function \( x^3 - x \) over the interval \([-1, 1]\).
(d) Explain these results graphically.
(e) Find the average value of an odd function \( f(x) \) over the interval \([-a, a]\). (Remember that \( f(x) \) is odd if \( f(-x) = -f(x) \).)
(f) Suppose now that \( f(x) \) is an even function (that is, \( f(-x) = f(x) \)) and its average value over the interval \([0, 1]\) is 2. Find its average value over the interval \([-1, 1]\).

**Exercise 4.24** The intensity of light cast by a street lamp at a distance \( x \) (in meters) along the street from the base of the lamp is found to be approximately \( I(x) = 400 - x^2 \) in arbitrary units for \(-20 < x < 20\).

(a) Find the average intensity of the light over the interval \(-5 < x < 5\).

(b) Find the average intensity over \(-7 < x < 7\).

(c) Find the value of \( b \) such that the average intensity over \([-b, b]\) is \( I_{av} = 10\).

**Exercise 4.25** Rates of hormone production and removal

Consider the rate of hormone production \( p(t) \) and the rate of removal \( r(t) \) given by

\[
p(t) = A(1 + \sin(\omega t)), \quad r(t) = A(1 + \cos(\omega t)).
\]

for \( \omega = \pi/12 \). Calculate the net increase in hormone over the time period \( 3 \leq t \leq 15 \) (in hours).

We find that

\[
H = \int_{3}^{15} [p(t) - r(t)] \, dt = \int_{3}^{15} [(1 + \sin(\omega t)) - (1 + \cos(\omega t))] \, dt
\]

\[
= \int_{3}^{15} (\sin(\omega t) - \cos(\omega t)) \, dt = \left( -\frac{\cos(\omega t)}{\omega} - \frac{\sin(\omega t)}{\omega} \right)_{3}^{15}
\]

\[
= \frac{-12}{\pi} \left( \cos \left( \frac{5\pi}{4} \right) + \sin \left( \frac{5\pi}{4} \right) - \cos \left( \frac{\pi}{4} \right) - \sin \left( \frac{\pi}{4} \right) \right)
\]

\[
= \frac{-12}{\pi} \left( -\cos \left( \frac{5\pi}{4} \right) - \sin \left( \frac{5\pi}{4} \right) - \cos \left( \frac{\pi}{4} \right) - \sin \left( \frac{\pi}{4} \right) \right)
\]

\[
= \frac{-12}{\pi} \left( -2 \sin \left( \frac{\pi}{4} \right) - 2 \sin \left( \frac{\pi}{4} \right) \right) = \frac{-12}{\pi} \left( -\frac{4}{\sqrt{2}} \right) = \frac{24\sqrt{2}}{\pi}
\]

We used the fact that \( 5\pi/4 \) is an angle in the third quadrant, so that \( \cos(5\pi/4) = -\cos(\pi/4) \) and \( \sin(5\pi/4) = -\sin(\pi/4) \).

Thus the amount of hormone that accumulated over the time interval \( 3 \leq t \leq 15 \), i.e. between 3:00 am and 3:00 pm is \( 24\sqrt{2}/\pi \).

**Exercise 4.26** The length of time from sunrise to sunset (in hours) \( t \) days after the Spring Equinox is given by

\[
l(t) = 12 + 4 \sin \left( \frac{\pi t}{182} \right)
\]
(a) Explain the meaning of the word “equinox” and describe what happens on that day according to the above formula.

(b) What is the length of the shortest and the longest day and when do these occur according to this formula?

(c) How long is one complete cycle in this expression?

(d) Sketch \( l(t) \) as a function of \( t \). 

(e) Find the average day-length over the month immediately following the equinox.

(f) Find the average day length over the whole year. Explain your result with a simple geometric or intuitive argument.

**Exercise 4.27** Consider the periodic function,

\[ f(t) = \sin(2t) + \cos(2t). \]

(a) What is the frequency, the amplitude, and the length of one cycle in this function? (You are asked to express \( f(t) \) in the form \( A \sin(\omega t + \varphi) \). Then we use the terminology \( A \) = amplitude, \( \varphi = \) phase shift, \( \omega = \) frequency.

(b) How would you define the average value of this function over one cycle?

(c) Compute this average value and show that it is zero. Now explain why this is true using a geometric argument.

**Exercise 4.28** The current in an AC electric circuit is given by

\[ I(t) = A \cos(\omega t) \]

The power in the circuit is defined as \( P(t) = I^2(t) \).

(a) What is meant by one cycle in this situation?

(b) Sketch graphs of \( I(t) \) and \( P(t) \). Explain why \( P(t) \) is always positive, and indicate how its zeros are related to zeros of \( I(t) \). What are the maximal and minimal values of each of these functions?

(c) Find the average power and the average current over half a cycle. (Note: in computing the average power, you will need to use the trick \( \cos^2(\omega t) = \frac{1}{2}(1 + \cos(2\omega t)) \)).

**Exercise 4.29** **Food surplus** Thousands of years ago, in the Fertile Crescent, the land was fertile and food production rate was plentiful and constant over time, but the population was growing steadily. Suppose that the rate of food production (in units of kg per year) is denoted \( P \), and that the population size was \( N(t) = N_0 e^{rt} \) where \( t \) is time in months and \( r \) is a per capita growth rate of the population. Assume that each person consumes food at the rate \( f \) kg/year. In year 0, it was realized that there was a food surplus, and a large mud-brick structure was built to store the surplus food. (The surplus food is any food left over after consumption by the population that year).
(a) Write down an expression that represents the rate of accumulation of food surplus (as a function of time). Use that expression to determine the net surplus, $S(t)$ at time $t = T$.

(b) At what year was the food production rate the same as the food consumption rate?

(c) When was the food surplus greatest?

(d) How large was the maximal stored surplus?
4.9 Solutions

Solution to 4.1
(a) At time $t \approx 1.1$.
(b) Whenever slopes are equal, e.g. at time $t \approx 3.3$.
(c) Car 1.
(d) Car 2.
(e) At time $t \approx 2.2$.

Solution to 4.2
(a) $a(t) = 4t + 5$
(b) $\frac{19}{6}$

Solution to 4.3
(a) $k = 1$
(b) $10(e^{-t} + t - 1)$

Solution to 4.4
(a) $v(t) = \frac{t^2}{2} - \frac{t^3}{3}$
(b) $\frac{1}{12}$

Solution to 4.5
(a) $23\frac{1}{3}$ km
(b) 32.5 l

Solution to 4.6 $\frac{143}{36}$

Solution to 4.7
(a) $\frac{34}{9}$ s
(b) $\frac{5375}{81}$ m

Solution to 4.8
(a) child2
(b) child1

Solution to 4.9 $A \left( \frac{23}{3} - \frac{3}{20} \cos(0.15) \right)$
Solution to 4.10 12000$

Solution to 4.11 7.5 mg

Solution to 4.12

(a) $t = 5\sqrt{2}h$  
(b) $\frac{100\sqrt{2}}{3} \approx 471.4 \cdot 10^3$ barrels

Solution to 4.13

(a) $t = 10$ h  
(b) $t_2 = 30$ h  
(c) $\frac{160}{3} \cdot 10^5$ gal

Solution to 4.14 $0.1 \left( \frac{765 + \frac{240}{\pi}}{\pi} \right) \approx 84.14$ t

Solution to 4.15

(a) $r = \ln 2$  
(b) $R(t) = 0.5e^{-t\ln 2}$  
(c) $\approx 0.7213$ rad

Solution to 4.16

(a) $\int_0^t I(s) - C(s) \, ds$  
(b) $\approx 1.5$ min  
(c) $\approx 3$ min  
(d) $\approx 2.5$ min

Solution to 4.17

(a) $Q(T) = Q(0) + \int_0^T I(t) - O(t) \, dt$  
(b) $A$ max., $B$ min. (see Figure 4.3)

Figure 4.3. Solution for problem 4.17.
4.9. Solutions

Solution to 4.18

(a) \[ P(t) = P_0 + \int_0^t I(s) - O(s) \, ds \]

(b) \( t \approx 2 \) greatest, \( t \approx 5 \) smallest

Solution to 4.19

(a) 18 cm

(b) \( \frac{18}{5} \) cm/day

Solution to 4.20

\( \bar{f} = \frac{2}{\pi} \)

Solution to 4.21

(a) \( \bar{f} = \frac{1}{n+1}, \) for \( n \neq -1. \)

(b) \( \bar{f} \to 0. \)

(c) area under curve decreases.

(d) \( \bar{g} = \frac{n}{n+1}, \) for \( n \neq -1; \bar{g} \to 1; \) area under curve approaches 1.

Solution to 4.22

(a) \( \bar{v} = c \left( 1 + \frac{e^{-t}}{t} - \frac{1}{t} \right) \) m/s

(b) \( \bar{v} \to \) cm/s

Solution to 4.23

(a) 0

(b) 0

(c) 0

(d) odd functions (symmetric about origin)

(e) 0

(f) 2

Solution to 4.24

(a) \( \bar{I} = \frac{1175}{3} \)

(b) \( \bar{I} = \frac{1151}{3} \)

(c) \( b = \sqrt{1170} \)

Solution to 4.25

\( 24 \frac{\sqrt{2}}{\pi} \)

Solution to 4.26

(a) Equal length of day and night, i.e. 12 h, and hence \( t = 0 \) (spring) or \( t = 182 \) (fall).

(b) Longest day has 16 h at \( t = 91; \) shortest day has 8 h at \( t = 273. \)
Figure 4.4. Solution for problem 4.26 (d)

(c) 364

(d) See Figure 4.4.

(e) \(12 + \frac{364}{15\pi} \left(1 - \cos \left(\frac{15\pi}{91}\right)\right) \approx 13 \, \text{h}.

(f) 12 \, \text{h} \text{ for } 364 \, \text{days}; 12.0009 \, \text{h} \text{ for } 365 \, \text{days.}

Solution to 4.27

(a) \(\varphi = \frac{\pi}{4}, A = \sqrt{2}, \omega = 2\)

(b) \(\bar{f} = \frac{1}{\pi} \int_0^\pi \sin(2t) + \cos(2t) \, dt\).

(c) \(\bar{f} = 0\); equal areas above and below x-axis.

Solution to 4.28

(a) \(P(t) = \frac{A^2}{2} (1 + \cos(2\omega t))\)

(b) Because \(P(t)\) is the square of \(I(t)\) it must be \(\geq 0\). \(I(t)\) and \(P(t)\) have the same zeroes. Maximal and minimal values: \(-A \leq I(t) \leq A\), \(0 \leq P(t) \leq A^2\).

(c) \(P = \frac{A^2}{2} \text{ W, } \bar{I} = 0 \text{ A.}\)

Solution to 4.29

(a) \(\frac{dS}{dt} = P - fN_0 e^{rt}, S(t) = Pt - \frac{t}{2} N_0 (e^{rt} - 1).\)
(b) \[ t = \frac{1}{r} \ln \left( \frac{P}{fN_0} \right) . \]

(c) \[ t_{\text{max}} = \frac{1}{r} \ln \left( \frac{P}{fN_0} \right) \text{ (see (b)).} \]

(d) \[ S_{\text{max}} = \frac{P}{r} \left( \ln \left( \frac{P}{fN_0} \right) - 1 \right) + \frac{f}{r} N_0. \]