This examination has 7 pages of questions excluding this cover

The University of British Columbia
Midterm 1 - February 4, 2013

Mathematics 103: Integral Calculus with Applications to Life Sciences

201 (Hauert), 203 (Hauert), 206 (Bruni), 207 (Maciejewski), 208 (Ronagh), 209 (Nguyen)

Closed book examination

Time: 60 minutes

Last Name: ___________________  First Name: ___________________

Student Number: ___________  Section: circle above

Rules governing examinations:
1. No books, notes, electronic devices or any papers are allowed. To do your scratch work, use the back of the examination booklet. Additional paper is available upon request.
2. You should be prepared to produce your library/AMS card upon request.
3. No student shall be permitted to enter the examination room after 10 minutes or to leave before the completion of the examination.
4. You are not allowed to communicate with other students during the examination. Students may not purposely view other’s written work nor purposefully expose his/her own work to the view of others or any imaging device.
5. At the end of the exam, you will put away all writing implements upon instruction. Students will continue to follow all of the above rules while the papers are being collected.
6. Students must follow all instructions provided by the invigilator.
7. Students are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
8. Any deviation from these rules will be treated as an academic misconduct. The plea of accident or forgetfulness shall not be received.

I agree to follow the rules outlined above

(signature)

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Important

1. Simplify all your answers as much as possible and express answers in terms of fractions or constants such as √e or ln(4) rather than decimals.
2. Show all your work and explain your reasonings clearly!

\[ \sum_{k=1}^{N} k = \frac{N(N+1)}{2}, \quad \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}, \quad \sum_{k=1}^{N} k^3 = \frac{N^2(N+1)^2}{4}, \]

\[ \sum_{k=0}^{N} r^k = \frac{1 - r^{N+1}}{1 - r} \quad (r \neq 1), \quad \int_{a}^{b} f(x) \, dx = \lim_{N \to \infty} \sum_{k=1}^{N} \Delta x f(x_k) \quad \left( \Delta x = \frac{b - a}{N}, x_k = a + k \Delta x \right) \]
1. (10 points) Summation - Short Answer Problems
   (full marks for correct answer; work must be shown for partial marks)

   a. Write the sum in summation (sigma) notation: \( S = \frac{2}{5} + \frac{4}{25} + \frac{6}{125} + \frac{8}{625} + \frac{10}{3125} \).

   ANSWER: \( S = \quad \) 

   b. Evaluate the sum: \( S = \sum_{k=0}^{9} (k + 1)^2 \).

   ANSWER: \( S = \quad \) 

   c. Evaluate the sum: \( S = \sum_{i=1}^{\infty} \frac{2^i}{6^{2i-1}} \).

   ANSWER: \( S = \quad \)
d. Evaluate the Riemann sum: \( R = \lim_{n \to \infty} \sum_{m=1}^{n} \frac{\sin \left( \frac{2\pi m}{n} \right)}{n} \).

**ANSWER:** \( R = \) 

e. Consider the Riemann sum: \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi}{8n} \tan \left( \frac{i \pi}{40n} \right) = \int_{a}^{b} f(x) \, dx. \)

Find \( f(x) \), \( a \), and \( b \). (Note: do not evaluate neither sum nor definite integral.)

**ANSWER:** \( a = \) \( b = \) \( f(x) = \)
2. (10 points) Definite Integrals - Short Answer Problems
   (full marks for correct answer; work must be shown for partial marks)

   a. Compute the integral \( I = \int_{-1}^{1} x^5 \, dx \) (or state that it does not exist).

   ANSWER: \( I = \) ________________________________

   b. Compute the integral \( I = \int_{-1}^{1} x^{-6} \, dx \) (or state that it does not exist).

   ANSWER: \( I = \) ________________________________

   c. Consider the function \( F(x) = \int_{x}^{\pi} e^{\cos t} \, dt \). Calculate the derivative \( F'(x) \).

   ANSWER: \( F'(x) = \) ________________________________
d. Compute the integral of \( I = \int_{0}^{3} \left( \sqrt{9 - x^2} + 2 \right) \, dx \) using high school geometry.

\[
I = \text{__________________________}
\]

e. Compute the average value \( \bar{f} \) of \( f(x) = |x^2 - 1| \) for \(-2 \leq x \leq 2\).

\[
\bar{f} = \text{__________________________}
\]
3. (8 points) Compute the sum $S = \sum_{n=1}^{20} \frac{2}{n^2 - n}$ and simplify as much as possible.

*Hint:* The identity $\frac{1}{n^2 - n} = \frac{1}{n - 1} - \frac{1}{n}$ may be useful.
4. (10 points) Calculate the finite area $A$ bounded by the two curves $f(x) = 1 - x^2$ and $g(x) = x - 1$. 
5. (12 points) Starting at time $t = 0$, an initially empty tank is filled with water at a rate $I(t) = e^{-2t+1}$ litres per second. Simultaneously, water is draining through a leak at the bottom. The leak is gradually getting clogged by debris and the water drains at a rate $R(t) = e^{-t}$ litres per second. The graphs of $I(t)$ and $R(t)$ are shown in the figure below.

a. Label the curves in the graph and clearly mark and label the (approximate) times when the water in the tank:
   i. is rising fastest
   ii. is draining fastest, and
   iii. is at its maximum.

b. Find the amount of water in the tank, $V(t)$, as a function of the time.

c. Calculate the time at which the amount of water reaches its maximum?

d. What is the minimum size of the tank required to ensure that it never overflows?

e. What is the average amount of water in the tank up to time $T$?

f. What is the long term average amount of water (i.e. as $T \to \infty$)?