\[ v(t) = -\int_0^t g \cdot e^{-\frac{kr}{2}} \, ds = -g \cdot e^{-\frac{kr}{2}} \left|_0^t \right. \]
\[ = \frac{g}{k} \left( e^{-\frac{kt}{2}} - 1 \right) \]

Important: we used \( v_0 = 0 \) to find const. in \( a(t) \)

\[ \rightarrow \text{cannot change } v_0 \text{ without adjusting } C \text{ in } a(t) \text{ too!} \]

> velocity converges to \( -\frac{g}{k} \) \( \rightarrow \) terminal velocity

> holds for any initial velocity

\[ x(t) = x(0) + \int_0^t v(s) \, ds = x_0 + \int_0^t \frac{g}{k} \left( e^{-\frac{kr}{2}} - 1 \right) \, ds \]

\[ a) \quad x(t) = x_0 - \frac{g}{k} t + \frac{g}{k^2} \left( 1 - e^{-\frac{kt}{2}} \right) \]

\[ b) \text{ max velocity: } v_\infty = -\frac{g}{k} \quad \rightarrow \text{terminal velocity}. \]
W: redo for $v_0 \neq 0$

4.2 Production & removal

$\rightarrow$ chemicals in body
$\rightarrow$ population size
$\rightarrow$ water level at Capilano reservoir

![Diagram showing reservoir](image)

how much water in reservoir?

1) for simple reservoir shapes: geometry: $V(t) = \text{area}_W \cdot h(t)$

2) natural reservoir?
   $\rightarrow$ keep track of influx $I(t)$ and outflux $O(t)$.
   $\rightarrow$ rate of change of water: $I(t) - O(t)$

$V(t) = V(0) + \int_{0}^{t} I(s) - O(s) \, ds$

$\rightarrow$ how to find $V(0)$? $\rightarrow$ start with empty reservoir.

![Graph showing reservoir levels](image)

- max. rate of increase
- max. rate of decrease

$\rightarrow$ how do $\int I(t) \, dt$ and $\int O(t) \, dt$ compare?
\[
\int I(t) \, dt = \int 0(t) \, dt \quad \text{otherwise}
\]

reservoirs is exist unsustainable.

4.3 Average of a function

\[\bar{f} \]

\[f(x)\]

Find average value of \( f(x) \)

\[\bar{f} \]

\[f \]

\[\bar{f} \]

is horizontal line such that \( f \) areas above and below are equal.

\[
\frac{1}{b-a} \int_{a}^{b} f(x) \, dx = \bar{f} \]

\[
\bar{f} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \quad \text{average of } f(x) \text{ over interval } [a, b]
\]

Ex. average height of stone (see earlier)

\[x(t) = V_0 t - \frac{g}{2} t^2 \quad V_0 = 50 \text{ m/s}, \quad T = \frac{2V_0}{g} \]

\[t\text{ time when stone is back on ground}\]

\[\bar{x} = \frac{1}{T} \int_{0}^{T} V_0 t - \frac{g}{2} t^2 \, dt \]

\[= \frac{1}{T} \left( \frac{V_0 T^2}{2} - \frac{g}{6} T^3 \right) \]

\[= \frac{V_0^2}{g} - \frac{8}{83} \frac{4V_0^2}{9g} = \frac{V_0^2}{3g} \quad (\approx 85 \text{ m})\]
$x_{max} \approx 127.4 \text{ m}$ (or $\frac{v_0^2}{2g}$)

4. b. Flu vaccination → course notes → work on your own.

Chapter 5: Mass, Volume & Length

5.1 Mass distributions

Consider a long iron cone

![Diagram of a cone with radius at base $r$ and density of iron $\rho$]

→ would cone into cylindrical rod of negligible mass.

![Diagram of a cylinder with radius $r$]

→ density of rod is function of $x$: $\rho(x)$ [kg/m$^3$]

$\rho(x) \approx \frac{\text{mass of thin slice}}{\text{volume of thin slice}} = \frac{\pi r(x)^2 \Delta x \cdot \rho}{r^2 \pi \cdot 2x} = \frac{\rho}{r^2} \cdot r(x)^2$

$
\left[ r(x) = \frac{r}{x} \cdot x \right] \text{ radius of cone at } x$

$= \frac{\rho \cdot x^2}{r^2} \cdot x^2 = \frac{\rho}{r^2} \cdot x^2$

a) Find total mass of rod

→ same as mass of cone

→ use geometry: $\frac{\frac{r^2}{2} \pi \cdot L \cdot \rho}{x} = M$

→ volume of cone.

→ using calculus:
\[ M = \int_0^a r^2 \pi s(x) \, dx = \frac{r^2 \pi}{2} \int_0^a x^2 \, dx \]

\[ = \frac{r^2 \pi}{2} \left[ \frac{x^3}{3} \right]_0^a = \frac{r^2 \pi}{6} a^3 \]

(as an aside: just calculated volume of cone!)

b) average density of rod

\[ \bar{s} = \frac{1}{l} \int_0^l s(x) \, dx = \frac{5}{l^3} \left[ \frac{x^3}{3} \right]_0^l = \frac{5}{3} \]

c) centre of mass

→ where to support cylindrical rod such that it is balanced?

Find \( \bar{x} \).

Excursions - Centre of mass

\[ m_i = \Delta x \cdot f(x) \] ; distance: \( \bar{x} - x \).
(2) \[ m_2 = \Delta x \cdot \bar{f}(x_2) \]; distance \( x_2 - \bar{x} \)

Moments pulling down on left must equal those pulling down at right. \( \Rightarrow \) balance

\[ \int_a^x (x-x) f(x) \, dx = \int_x^\bar{x} (x-x) f(x) \, dx \]

\[ \int_a^\bar{x} (\bar{x}-x) f(x) \, dx - \int_a^x (x-x) f(x) \, dx = 0 \]

\[ \int_a^\bar{x} (\bar{x}-x) f(x) \, dx + \int_x^b (x-x) f(x) \, dx = \int_a^\bar{x} (x-x) f(x) \, dx = 0 \]

\[ \bar{x} \int_a f(x) \, dx - \int_a^x x f(x) \, dx = 0 \]

\[ \bar{x} = \frac{\int_a^b x f(x) \, dx}{\int_a f(x) \, dx} \]

(we will return to this in ch. 9 when talking about the mean of probability distrib.)

\[ \Rightarrow \text{apply to rod: } S(x) = 5 \frac{x^2}{l^2} \]

\[ \bar{x} = \frac{\int_0^l x S(x) \, dx}{\int_0^l S(x) \, dx} = \frac{5}{l^2} \int_0^l \frac{x^4}{4} \, dx = \frac{5}{4} \frac{l^4}{4} = \frac{5}{4} \frac{l^4}{3} \]

we cancel out the \( l \) terms...

\[ \frac{5}{3} \]
d) cut rod into two pieces of equal weight

\[ S(x) \]

\[ x_{1/2} \] (point where we have same mass on left and right)

Excursion - find \( x_{1/2} \)

\[ f(x) \]

\( x_{1/2} \) such that

\[ A_1 = A_2 = \frac{A}{2} \]

\[ \int_{x_{1/2}}^{b} f(x) \, dx = \frac{1}{2} \int_{a}^{b} f(x) \, dx \]

\( \rightarrow \) calculate integral, solve for \( x_{1/2} \)

\( \rightarrow \) apply to rod:

\[ \int_{0}^{x_{1/2}} S(x) \, dx = \frac{1}{2} \int_{0}^{b} S(x) \, dx = \frac{S}{6} l \]

\[ \frac{S}{6} x^3 \bigg|_{0}^{x_{1/2}} = \frac{S}{6} \frac{3}{2} l \]

\( l = 0.8 \)

\[ x_{1/2} = \frac{3}{2} \]

\[ x_{1/2} = \sqrt[3]{\frac{3}{2}} l \] (not)
if mass distribution is symmetric about \( \bar{x} \) then
\[ \bar{x} = \bar{x} \sqrt{2} \]
otherwise \( \bar{x} \) is shifted to side with longer tail.

5.2 Application: high blood pressure

- increase of blood pressure in constricted blood vessels

\[ \Delta P = P_H - P_L \]

Blood vessel: \( l \): length, \( \Delta P \): pressure difference, \( \eta \): viscosity of blood

- velocity: \( v(r) = \frac{\Delta P}{4\eta l} (R^2 - r^2) \)
  (from experiments)

- find flux of blood (amount of blood through vessel per time per area)

Cross-section of vessel

- same velocity along ring

- find area of ring
\[ \Delta A = (r + dr)^2 \pi - r^2 \pi \]
\[ = (r^2 + 2rdr + (dr)^2) \pi - r^2 \pi \]
\[ = 2rdr \pi + (dr)^2 \pi \]

- for small \( dr \): \( (dr)^2 \ll dr \)
  \( \uparrow \) much smaller
\[ \Rightarrow \text{approx. area of thin ring } a_0 \approx 2\pi r \, dr \] 

\[ \Rightarrow \text{flux through ring: } a_0(r) \cdot v(r) \] 

\[ \Rightarrow \text{flux through vessel:} \]

\[ F(R) = \int_0^R 2\pi R \, \frac{\Delta P}{2\pi \eta l} (R^2 - r^2) \, dr = \frac{\Delta P \pi}{2 \eta l} \left( \frac{r^2}{2} - \frac{r^4}{4} \right) \bigg|_0^R \]

\[ = \frac{\Delta P \pi}{2 \eta l} \left( \frac{R^4}{2} - \frac{R^4}{4} \right) = \frac{\Delta P \pi}{8 \eta l} R^4 \]

**Ex.** If radius is decreased to \(\frac{3}{4} R\) what increase in \(\Delta P\) is needed to maintain the same flux.