\[ \frac{dN}{dt} = xN - \frac{r N^2}{K} \]

- reproduction
- competition

- changing and changes both reproduction & competition
- unclear biological reasons

- \( r < 0 \): death rates > birth rates
- biologically non-sense for \( N > K \)

### 9.3 Other examples

**Ex 1)** \( \frac{dy}{dx} = y \cdot x \vee y(2) = 1 \)

1. Separate variables:
   \[ \frac{1}{y} \ dy = \frac{1}{x} \ dx \]

2. Anti-derivative:
   \[ \ln |y| = \frac{x^2}{2} + C \]

3. Solve for \( y \):
   \[ y(x) = e^{\frac{x^2}{2}} \cdot C \]

   Find \( C \):
   \[ y(2) = e^2 \cdot C = 1 \rightarrow C = e^{-2} \]

   \[ y(x) = e^{\frac{x^2}{2} - 4} \]

**Notes:**
- any steps 0-3 may not be possible
- numerical solutions (Euler's method)
- equilil:
  \[ \frac{dy}{dx} = 0: \ y = 0 \text{ or } x = 0 \]
  \( \text{indep. variable} \)
  1. Potential equilil: \( \rightarrow \) not equililibrum!
Ex 2) \( \frac{dy}{dx} = x - y \); \( y(2) = 1 \)

→ separation is not possible
→ try to guess! \( y(x) = x - 1 \)
→ verify potential solution is easy! → plug-in.
LHS: \( \frac{dy}{dx} = 1 \); RHS: \( x - x + 1 = 1 \) DE ok.
initial cond. ok. → \( y(x) = x - 1 \) is indeed sol.
→ solution does not work for \( y(2) = 0 \).
→ another guess: \( y(x) = C \cdot e^x + x - 1 \)
find \( C \): \( y(2) = C \cdot e^2 + 1 = 0 \) → \( C = -e^2 \)
→ check that \( C = -e^2 \) is sol. of DE.

Chapter 10: Sequences

A sequence is ...

\( (1, 2, 3, 4, 5, \ldots) \)  
continue with pattern outlined by first terms

\( (1, 1, 2, 3, 5, 8, 13, \ldots) \)  
find rule that generates sequence  
(Fibonacci sequence)

\( (0, 1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots) \)  
not just integers

Notation: \( \sum a_k \) dummy variable  
\( a_k \geq 0 \) starting value (does not have to be zero)
or \((a_k)_{k=0}^{\infty}\) or \((a_k)_{k=0}^{\infty}\)

\[
\rightarrow \text{represents: } (a_0, a_1, a_2, a_3, \ldots)
\]

\[
\rightarrow \text{http://oeis.org.}
\]

- closed form: \(a_k = k^2 - 2^k + 1 = f(k)\)

\[
\rightarrow \text{evaluated at integer } k
\]

- recursive relations:
  
  Ex. 1) \(a_{k+1} = \frac{1}{2} a_k\) with \(a_0 = 1\)

  \[
  \rightarrow (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)
  \]

  \[
  \rightarrow \text{closed form: } a_k = \left(\frac{1}{2}\right)^k
  \]

  Ex. 2) \(a_{k+1} = a_k + a_{k-1}\) with \(a_0 = a_1 = 1\)

  \[
  \rightarrow (1, 1, 2, 3, 5, \ldots)
  \]

  \[
  \rightarrow \text{Fibonacci}
  \]

- iterated maps (special kind of recursion)
  
  \[a_{k+1} = f(a_k) = f(f(a_{k-1}))\] with \(a_0\) given

\[
\rightarrow \text{enables us to write sequences in simple compact form.}
\]

What does sequence do for large \(k\)?

\[
\rightarrow \text{does it converge to some (finite) value } L?
\]

\[
\rightarrow \lim_{k \to \infty} a_k = L
\]

\[
\rightarrow \text{if limit does not exist, the sequence diverges}
\]

(\text{does not converge})
Ex. 1) $\sum_{k=0}^{\infty} \frac{1}{k+1} = (1, 1, 1, 1, \ldots) \\
\rightarrow$ const. sequence, converges

2) $\sum_{k=1}^{\infty} \frac{1}{k} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots) \\
\rightarrow$ harmonic sequence converges \( (\text{limit is } 0) \)

3) $\sum_{k=1}^{\infty} \frac{1}{k!} = (1, 1, 2, 6, \ldots) \\
factorial: k! = k(k-1)(k-2)\ldots 3 \cdot 2 \cdot 1; 0! = 1 \\
\rightarrow$ subsequent terms get bigger \\
$\rightarrow$ diverges (approaches $+\infty$)

4) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = (0, -1, 2, -3, 4, \ldots) \\
\rightarrow$ magnitude increases but signs oscillate \\
$\rightarrow$ diverges

5) $\sum_{k=1}^{\infty} \frac{\cos(\pi/k)}{k} = (1, 0, -1, 0, 1, \ldots) \\
\rightarrow$ keeps oscillating (alternating) \\
$\rightarrow$ does not converge, diverges \\
$\rightarrow$ bounded sequence

A sequence is bounded if $|a_k| \leq M$ for finite, fixed $M$

A sequence is monotone

1) $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_{k-1} \leq a_k \leq \cdots \\
\rightarrow$ monotone increasing

2) $a_0 \geq a_1 \geq a_2 \geq \cdots \Rightarrow a_{k-1} \geq a_k \geq \cdots \\
\rightarrow$ monotone decreasing
Ex. geometric sequence: $\{r^k\}_{k=0}^\infty = (1, r, r^2, r^3, \ldots)$

- $r > 1$: monotone increasing, unbounded, diverging ($+\infty$)
- $r < -1$: diverges
- $r = 1$: HV.
- $0 < r < 1$: converges
- $r = 0$: $0$ for all $k$
- $0 > r > -1$: converges

Note: convergent vs. bounded sequences
- if $\sum_{k=0}^\infty a_k$ converges → sequence is bounded
- if $\|a_k\|$ unbounded → sequence diverges

→ the reverse does not hold.

10.1 Limits of Sequences and Convergence

for sequences $\{a_k\}_{k=0}^\infty$ in closed form $a_k = f(k)$

we can use limit $f(k)$ and check if it exists $k \to \infty$

and what the limit is.

→ taking limit implies that $f(x)$ with $x \in \mathbb{R}$
   (not just integers)

→ may not work for all functions

Ex. 1) $f(k) = \left(-\frac{1}{2}\right)^k \cdot k$

→ closed form, ok for $k \in \mathbb{N}$ (integers)
→ undefined for arbitrary $k$

2) $f(k) = \frac{k!}{k^k}$

→ $k!$ defined for integer $k$ → extension to $\mathbb{R}$ interactively
Squeeze Theorem: if $b_k \leq a_k \leq u_k$ for $k > K$

and $\lim_{k \to \infty} b_k = L = \lim_{k \to \infty} u_k$

then $\lim_{k \to \infty} a_k = L$.

→ apply to prev. examples:

1) $a_k = \left(-\frac{1}{2}\right)^k \cdot k$

   $b_k = -\left(\frac{1}{2}\right)^k \cdot k \implies \lim_{k \to \infty} \left(-\frac{1}{2}\right)^k \cdot k$

   $\implies \lim_{k \to \infty} \frac{k}{2^k} = 0$

   $\implies \lim_{k \to \infty} a_k > 0$

   $\implies \lim_{k \to \infty} u_k = 0$

2) $a_k = \frac{k!}{k^k} = \frac{k}{k} \cdot \frac{(k-1)}{k} \cdot \frac{(k-2)}{k} \cdot \ldots \cdot \frac{2}{k} \cdot \frac{1}{k}$

   $\implies a_k$ must be smaller than any particular term.
   → pick one as upper bound

   $u_k = \frac{1}{k} \geq a_k \implies \lim_{k \to \infty} \frac{1}{k} = 0$

→ for $k > 0$ we have $a_k \geq 0$ so use $b_k = 0$

→ sequence converges; $\lim_{k \to \infty} a_k = 0$
Note: if lower bound \( \to \infty \) then \( \lim_{x \to \infty} a_x \to \infty \)
if upper bound \( \to -\infty \) then \( \lim_{x \to -\infty} a_x \to -\infty \)

10.2. Applications of sequences

A) Newton's method - solve \( f(x) = 0 \) numerically

\[ L_0(x) = f(x_0) + f'(x_0)(x-x_0) \]

- Tangent line at \( x_0 \): \( L_k(x) = f(x_k) + f'(x_k)(x-x_k) \)
- \( x_{k+1} \): solve \( L_k(x) = 0 \)

\[ x = x_k - \frac{f(x_k)}{f'(x_k)} \]

\( \Rightarrow \) approximations \( (x_0, x_1, x_2, x_3 \ldots) \) represent a sequence. If it converges it approx. root of \( f(x) \).

\( \Rightarrow \) at limit we have \( x_{k+1} = x_k \) which implies that \( f(x_k) = 0 \).

B) Iterated maps: \( a_2 = f(a_1) = f(f(a_0)) = \ldots \) = \( f^k(a_0) \) - apply map \( f \) \( k \) times
1) graphical approach - cobwebbing

\[ y = x \]

\[ x_{k+1} = f(x_k) \]

**Procedure:**

1. Choose start \( x_0 \)
2. Find \( x_{k+1} = f(x_k) \) (vertical)
3. Map \( x_{k+1} \) onto horizontal axis
   \[ \rightarrow \text{intersection with diagonal} \]
   \[ \text{(horizontal)} \]
4. Continue with 2)

\[ \rightarrow \text{sequence converges to} \ x^* : \ x^* = f(x^*) \]
\[ \rightarrow \ x^* \text{ is fixed point / equilibrium of map } f(x) \]

- Fixed points of iterated maps are intersections of \( f(x) \) with diagonal line (solve \( x = f(x) \))
- Stability: a fixed point is stable if sequence \( x_{k+1} = f(x_k) \) converges to \( x^* \) for \( x_0 \) close to \( x^* \)

1) graphically!
2) Formal approach
   - Find equilibrium: solve $x = f(x)$
     \[ x^* \text{ is mapped onto itself} \]
   - Determine stability:
     \[ x_{t+1} = f(x_t) \]
     \[ x^* + \Delta x \approx f(x^*) + \Delta x \cdot f'(x^*) \]
     Distance from $x^*$
     Linear approximation at $x^*$
     \[ \Delta x_{t+1} \approx \Delta x \cdot f'(x^*) \]
     \[ |\Delta x_{t+1}| < |\Delta x| \text{ if } |f'(x^*)| < 1 \]
   - $x^*$ is stable if $\left| \frac{df}{dx} \right|_{x=x^*} < 1$
   - $x^*$ is unstable if $\left| \frac{df}{dx} \right|_{x=x^*} > 1$
   - Neutral stability if equal to 1

Note: $\frac{df}{dx} \bigg|_{x=x^*} < 0$ sequence is alternating near $x^*$
$\frac{df}{dx} \bigg|_{x=x^*} > 0$ is monotonous
Ex. \( x_{n+1} = f(x_n) = x_n^4 - 4x_n^2 + 2 \)

Equilibria: \( x = f(x) \)

- \( x_1^* = 2 \), \( x_2^* = -1 \)
- results in quadratic poly.
- first last two equil.

Stability:
- cobwebs
- verify formally

⇒ all equil. unstable.

\[ \text{Note: } x = x^4 - 4x^2 + 2 \Rightarrow x^4 - 4x^2 - x + 2 = 0 \Rightarrow (x-2)(x+1)(x^2-x-1) = 0 \Rightarrow \text{last two equil.} \]

10.3 Logistic Map

\[ N_{t+1} = \alpha \cdot N_t - \beta \cdot N_t^2 \]

- rule of reproduction
- competition

\( N_e \): size of pop.

Illustration:

\[ \frac{N_{t+1}}{N_t} = \alpha - \beta \frac{N_t}{K} \]

- avg. net rate of reproduction of each individual

- \( \frac{N_{t+1}}{N_t} = 0 \) pop. collapses (\( N_e < 0 \) biologically meaningless)

- if \( \frac{N_{t+1}}{N_t} = 1 \) pop. stays const. → equilibrium → carrying capacity \( K \).
If \( N_t < K \) then \( \frac{N_{t+1}}{N_t} > 1 \) → pop. grows

If \( N_t > K \) then \( \frac{N_{t+1}}{N_t} < 1 \) → pop. shrinks

→ can we conclude that \( K \) is stable?

→ not possible

Simplify map: transform vars: \( X_t = \frac{N_t}{K} \)

\[ X_{t+1} = aX_t(1-X_t) \quad \text{with} \quad 0 \leq X_t \leq 1 \]

Equilibria: \( X = aX(1-X) \) → \( X^*_0 = 0 \)

\[ X^*_1 = 1 - \frac{1}{a} \]

Stability: \( \frac{d}{dx} (x \cdot x(1-x)) \bigg|_{x=x^*} = 2a - 2aX^* \)

\( X^*_0 = 0 \): is stable if \( 0 < a < 1 \) → extinction

\( X^*_1 \) does not exist

Dynamics: \( 0 < a < 1 \): \( X^*_0 \) is stable extinction

\( 1 < a < 3 \): \( X^*_1 \) appears; stability: stable if \( 3 < a < 1 \)

→ becomes unstable for \( a > 3 \)

\( a > 3 \): fixed point replaced by cycle of period 2

\( (a \approx 3.1) \)

→ cycle of period 2 replaced by period 4 \( (a \approx 3.5) \)

→ 4, 8 \( (a \approx 3.8) \)

→ happens faster and faster

→ chaos for \( a_{\text{chaos}} \approx 3.57 \) (periods of any length)

→ "period doubling route to chaos"
Simple dynamics may again occur for:

\( x > x_c \): chaos, period 3 for \( x = 3.83 \)

\( 4 > x \): population collapses

Plot of attractors (cycles) as a function of \( x \):

- Attractor: states that are repeatedly visited (e.g., as part of a cycle)

⇒ check out links to movie and online coding tools (see wiki).