important difference between definite integrals and the area under a curve

Ex. Area between curves

Find area between \( f(x) = -x^2 + 2x \) and \( g(x) = x \)

\( A = \int_a^b [f(x) - g(x)] \, dx \)

\( A \) is only finite area between \( f \) and \( g \)

\( \Rightarrow \) how to calculate \( A \)?

\( \Rightarrow \) boundaries of integral: \( f(x) = g(x) \)

\( -x(x-2) = x \rightarrow x_1 = 0 < a \)

\( + (x-2) = 1 \rightarrow x_2 = 1 < b \)

\( \Rightarrow \int_0^1 [f(x) - g(x)] \, dx = \int_0^1 -x^2 + x \, dx \)

\( = \text{Riemann sums} = \frac{1}{6} \quad \text{by Eq!} \)
chapter 3: The fundamental theorem of calculus (FTC)

3.1 The definite integral

\[ \int_a^b f(x) \, dx = \lim_{N \to \infty} \sum_{k=1}^N f(x_k) \Delta x \]

- \( \Delta x \): width of strips \( N \to \infty, \Delta x \to 0 \)
  \( \Rightarrow \Delta x \): width of infinitesimally thin strip
- bounds \( a, b \) are hidden in \( \Delta x \) and \( x_k \)
- \( x \) is another dummy variable (result does not depend on it)
  \[ \Rightarrow \int_a^b f(x) \, dx = \int_a^\xi f(s) \, ds \]

- \( f(s) \) must be well-behaved in \([a, b]\)
  - defined
    \[ \Rightarrow \text{counter ex. : } \sqrt{x} \text{ for } x < 0 \]
  - bounded (remain finite)
    \[ \Rightarrow \text{counter ex. : } \frac{1}{x^2} \text{ for } x \to 0 \]
  - continuous
    \[ \Rightarrow \text{counter ex. : } f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases} \]
    \[ \Rightarrow \text{no jumps} \]
  \[ \Rightarrow \text{for } x < a \text{ or } x > b \ f(x) \text{ can be nasty.} \]
3.2 Properties of definite integrals

1) \[ \int_{a}^{b} f(x) \, dx = 0 \]

Note: also holds for \( c > b \) or \( c < a \)!

(requires that \( f(x) \) is well behaved over larger integral)

2) \[ \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \]

3) \[ \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx = \int_{a}^{b} [f(x) + g(x)] \, dx \]

4) \[ c \int_{a}^{b} f(x) \, dx = \int_{a}^{b} cf(x) \, dx \]

\( C \) : constant
5) \[ \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \]

- area should not depend on whether we start at \( a \) and end in \( b \) or vice versa but definite integral changes sign.
- reason: \( \Delta x < 0 \) when starting at upper bound.

3.3 Areas and definite integrals

a) \[ \int_{a}^{b} f(x) \, dx \]
\[ A = \int_{a}^{b} f(x) \, dx > 0 \]

b) \[ \int_{a}^{b} f(x) \, dx \]
\[ \int_{a}^{b} f(x) \, dx < 0 \]
\[ \rightarrow A = -\int_{a}^{b} f(x) \, dx \]

or \[ \int_{a}^{b} |f(x)| \, dx \]
absolute value

\[ \int_{a}^{b} f(x) \, dx = ? \]

\[ A = A_1 + A_2 \]

\[ \int_{a}^{b} f(x) \, dx = ? \]
\[ A_1 = \int_a^k f(x) \, dx > 0, \quad A_2 = -\int_c^x f(x) \, dx < 0 \]

\[ \int_a^k f(x) \, dx = A_1 - A_2 \]

Note: regions with \( f(x) < 0 \) make negative contributions to definite integral.

Overall area determined by adding pieces where \( f(x) > 0 \) and subtracting those where \( f(x) < 0 \)

\[ \text{see ex. } \int x^3 - 3x^2 + 2x \, dx \text{ from chapter 2.} \]

In W&V3: 'signed area' \( \rightarrow \) interpret as 'definite integral'

3.4 Area as a function

\[ A(x) = \int_a^x f(x) \, dx \]

\( \rightarrow \) bad notation!

what is \( x \)?

\( \rightarrow \) instead: \( A(s) = \int_a^s f(x) \, dx \) or \( A(x) = \int_a^x f(s) \, ds \)

How does \( A(x) \) change if \( x \) is slightly increased?

\[ A(x+\Delta x) = \int_a^{x+\Delta x} f(s) \, ds \]

\[ = \int_a^{x+\Delta x} f(s) \, ds + \int_{x+\Delta x}^x f(s) \, ds \]
\[
\begin{align*}
A(x) &= A(x) + \int_{x}^{x+\Delta x} f(s) \, ds \\
&\approx \Delta x \cdot f(x) \\
&\approx \Delta x \cdot f(x+\Delta x) \\
\Rightarrow \text{ change in } A(x): A(x+\Delta x) - A(x) &\approx \Delta x \cdot f(x) \\
\frac{A(x+\Delta x) - A(x)}{\Delta x} &\approx f(x) \\
\text{approx. improves for } \Delta x \to 0
\end{align*}
\]

\[
\lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x} = f(x) = \frac{dA(x)}{dx} \quad (= A'(x))
\]

definition of derivative

\[
\Rightarrow A(x) = \int_{a}^{x} f(s) \, ds \quad \Rightarrow f(x) = \frac{dA(x)}{dx}
\]

\[\text{deep connection between differential and integral calculus}\]

\[\text{essence of FTC}\]

3.5. FTC

3.5.1 Part I

Let \( f(s) \) be bounded and continuous in interval \([a, b]\) and

\[
A(x) = \int_{a}^{x} f(s) \, ds
\]

Then for \( a \leq x \leq b \)

\[
\frac{dA}{dx} = f(x)
\]
Proof: see prev. section.

Recall derivatives:

\[ g_1(x) = x^3 \Rightarrow \frac{dg_1}{dx} = 3x^2 \]
\[ g_2(x) = x^3 + 5 \Rightarrow \frac{dg_2}{dx} = 3x^2 \]

\[ g(x) = f(x) + C \quad \text{\( C \): const (does not depend on \( x \))} \]

\[ \frac{dg}{dx} = \frac{df}{dx} \]

\[ \rightarrow \text{adding const to fun. does not change its derivative} \]

\[ \text{slope is derivative at a } f'(a) \]

Explore anti-derivatives \( \rightarrow \) reverse

\[ f_1'(x) = 3x^2 \Rightarrow f_1(x) = \frac{x^3}{3} \]
\[ f_2'(x) = 2x^2 \Rightarrow f_2(x) = \frac{x^3}{3} + 5 \]

\[ \rightarrow \text{anti-derivatives only defined up to a constant} \]

\[ f'(x) \rightarrow f(x) + C \quad \text{(=: F(x))} \]

\[ \text{define } F(x) \text{ as \text{anti-derivative of } f(x)} \]
3.5.2 FTC - part II

Let \( f(x) \) be bounded and continuous over interval \([a, b]\).
Suppose \( F(x) \) is any anti-derivative of \( f(x) \) then
\[
A(x) = \int_a^x f(s) \, ds = F(x) - F(a)
\]

Proof: \( A(x) = \int_a^x f(s) \, ds = F(x) + C \)

part I: \( A(x) \) is anti-derivative of \( f(x) \)
so is \( F(x) \)
\[
A(a) = \int_a^a f(s) \, ds = F(a) + C = 0
\]
\[\rightarrow C = -F(a)\]

\[\rightarrow \text{plug in: } A(x) = \int_a^x f(s) \, ds = F(x) - F(a) \]

\[\rightarrow \text{important tool to determine definite integrals}\]
\[\rightarrow \text{no more Riemann sums}\]

3.6 Review of (anti-) derivatives

<table>
<thead>
<tr>
<th>function</th>
<th>derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^n )</td>
<td>( n \cdot x^{n-1} )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>( \cos x )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( -\sin x )</td>
</tr>
<tr>
<td>( e^x )</td>
<td>( e^x )</td>
</tr>
<tr>
<td>( \ln x )</td>
<td>( \frac{1}{x} )</td>
</tr>
</tbody>
</table>

anti-derivatives ← functions
(plus constant)
3.7 Applications of FTC

Ex. 1) Find \( A = \int_0^1 (x+1) \, dx \)

a) geometry: \( A = \frac{3}{2} \).

- perfectly fine way to solve definite integral.

b) \( \int_0^1 (x+1) \, dx = \int_0^1 dx + \int_0^1 x \, dx \)

\[ 0 \quad \frac{d}{dx} \quad 1 \]

\( 0 \quad \frac{d}{dx} \quad 1 \)

\[ \int_0^1 dx = (x+C) \Big|_0^1 = 1+C - 0 - C = 1 \]

Notation: \( F(x) \bigg|_a^b = F(b) - F(a) \)

definition

\[ \int_0^1 x \, dx = \frac{1}{2} x^2 + C \Big|_0^1 = \frac{1}{2} + C - C = \frac{1}{2} \]

0 + 2: \( \frac{3}{2} = A \checkmark \)

- in definite integrals the integration constant never shows up.

2) \( A = \int_{-1}^1 e^{-2x} \, dx = \frac{1}{2} e^{-2x} \Big|_{-1}^1 = \frac{1}{2} (e^{-2} - e^2) \)

- easy to check anti-derivatives by taking derivative
3) a) Calculate \( \int_{-2}^{2} x^3 - 4x \, dx \)

\[ \int_{-2}^{2} x^3 - 4x \, dx = \left( \frac{1}{4} x^4 - 2x^2 \right) \bigg|_{-2}^{2} = 4 - 8 \quad \text{\color{red}{0}} \quad 4 + 8 = 0 \quad \text{\color{red}{0}} \quad \text{careful with signs} \]

\[ \Rightarrow A_1 = A_2 \]

b) \( \int_{-2}^{0} x^3 - 4x \, dx - \int_{0}^{2} x^3 - 4x \, dx = -2 \int_{0}^{2} x^3 - 4x \, dx \)

\[ = -2 \left( \frac{x^4}{4} - 2x^2 \right) \bigg|_{0}^{2} = -8 + 16 = 8 \]

Note: \( f(x) = x^3 - 4x \) is an odd function.

Recall: \( f(x) \) is odd if \( f(-x) = -f(x) \)

(point symmetry about origin)

Consider: \( \int_{a}^{b} f(x) \, dx \) with \( f(x) \) is odd (\( a \geq 0 \))

\[ \int_{-a}^{a} f(x) \, dx = \int_{-a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx \]

\[ \Rightarrow \quad x \rightarrow -x : \text{changes signs of boundaries and direction of integration} \]

\[ = -\int_{-a}^{0} f(-x) \, dx + \int_{0}^{a} f(x) \, dx \]

\( f(x) \) is odd:

\[ \int_{a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx = -\int_{a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx = 0 \]
The integral over an odd function with symmetric interval about zero is zero.

Ex 4) \( \int_{-\pi}^{\pi} \sin\left(\frac{x}{2}\right) \, dx = 0 \) (\( \sin x \) is odd function, interval symmetric about zero)

What about even functions?

If \( f(x) \) is even, then \( f(x) = f(-x) \) (reflection along y-axis)

\[
\int_{-a}^{a} f(x) \, dx = \int_{0}^{a} f(x) \, dx + \int_{0}^{-a} f(x) \, dx
\]

\( \quad \xrightarrow{x \to -x} \)

\[= -\int_{-a}^{0} f(-x) \, dx + \int_{0}^{a} f(x) \, dx = -\int_{a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx
\]

\[= 2 \int_{0}^{a} f(x) \, dx
\]

Ex 5) \( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx = 2 \int_{0}^{\frac{\pi}{2}} \cos x \, dx = 2 \cdot \sin x \bigg|_{0}^{\frac{\pi}{2}} = 2 \)