If distribution is symmetric about $x$ then $\bar{x} = x_{1/2}$

otherwise $\bar{x}$ shifted towards the tail

5.2 Application: High blood pressure

Blood vessel: length $L$, radius $R$, velocity $V(r)$

viscosity $\eta$

$V(r) = \frac{\Delta P}{4\eta L}$ from physics experiments

→ find flux of blood through vessel
(amount per time and per area)

cross section

→ velocity is same in ring

area of thin ring

$A_0(r) = \pi(r + dr)^2 - \pi r^2 = 2\pi r dr + dr^2 \pi$

→ for small $dr$: $dr \gg dr^2$

much bigger

$A_0(r) \approx 2\pi r dr \pi$ (circumference * thickness)
flux through ring: \( \mathbf{v}(r) \cdot \mathbf{a}_0(r) \)

flux through vessel:

\[
F(R) = \int_0^R 2\pi r \mathbf{v} \cdot \mathbf{a}_0 (r^2 - r^2) dr = \frac{\pi \Delta P}{2\eta l} \left( \frac{R^4 - r^4}{2} \right) \bigg|_0^R = \frac{\pi \Delta P}{2\eta l} \frac{R^4}{2} = \frac{\pi \Delta P}{8\eta l}
\]

Poiseille's law

Ex. If radius is decreased by \( \frac{3}{4} R \) what increase in pressure difference \( \Delta P \) is needed to maintain the same flux?

\[
F(R) = F\left(\frac{3}{4} R\right)
\]

for \( \Delta P \) for \( x \cdot \Delta P \) \( x \) times increase in pressure diff.

\[
x = \left(\frac{4}{3}\right)^4 \approx 3.2
\]

(HW)

\[
\text{reducing radius to } \frac{3}{4} R \text{ requires a 3.2-fold increase in } \Delta P \text{ to maintain the same flux.}
\]

5.3 Volumes – solids of revolution

for \( f(x) \) rotate \( f(x) \) around \( x \)-axis

\( \rightarrow \) find volume

1) cut thin slices

Volume of slice

\[
V(x) = \pi (f(x))^2 \Delta x
\]

2) add slices: \( V = \int_a^b \pi (f(x))^2 \, dx \)
Ex. Volume of sphere

semi-circle: \( y = f(x) = \sqrt{r^2 - x^2} \)

geometry: \( V = \frac{4}{3} \pi r^3 \)

using calculus: \( V = \int_{-r}^{r} f(x)^2 \, dx = 2\pi \int_{0}^{r} r^2 - x^2 \, dx \)

\[ = 2\pi \left[ r^2 x - \frac{x^3}{2} \right]_{0}^{r} = 2\pi \left( r^3 - \frac{r^3}{3} \right) = \frac{4\pi}{3} r^3 \]

Ex. Consider \( f(x) = a^2 - x^2 \)

First volume when rotating \( f(x) \) about different axes

a) rotate around \( x \)-axis from \( x = -a \) to \( x = a \)

\[ V = \int_{-a}^{a} \pi \left( a^2 - x^2 \right)^2 \, dx = 2\pi \int_{0}^{a} a^4 - 2a^2 x^2 + x^4 \, dx \]

\[ = 2\pi \left[ a^4 x - \frac{2a^2 x^3}{3} + \frac{x^5}{5} \right]_{0}^{a} = \ldots = \frac{16}{15} \pi a^5 \]

football or lemon
2) Rotate around y-axis for $0 \leq x \leq a$

\[ y = f(x) = a^2 - x^2 \]

\[ V = \pi \int_0^a (a^2 - y)^2 dy = \pi \int_0^a (a^2 - (a^2 - x^2))^2 dy \]

\[ = \pi \left( a^4 - \frac{a^4}{2} \right) = \frac{\pi}{2} a^4 \]

3) Rotate around $y = -c$ for $-a \leq x \leq a$

\[ V = \pi \int_{-a}^a (f(x) + c)^2 dx = \frac{8}{15} a^5 + \frac{4}{3} a^3 c + ac^2 \]

| HW |

- Length of strips increases by $c$
- Radius of disk
- Shift function up by $c$
- Rotate around x-axis

- Lemon with tips cut off.
(a) Rotate around \( x = d \) for \( d \leq x \leq a \\

\[ V = \pi \int_{0}^{a} d \cdot f(x)^2 \, dx \]

---

(b) \( f(x) \rightarrow f(x+d) = a^2 - (x+d)^2 \)

---

(c) Rotate the function \( f(x) \) to the left and rotate around the \( y \)-axis.

\[ f(x) \rightarrow f(x+d) = a^2 - (x+d)^2 \]

---

(e) vartient for rotating around \( y \)-axis

---

Use shells for part d)

---

Use shells for part d)
\[ V = \int_{0}^{a} 2\pi x \left( a^2 - (x + d)^2 \right) dx = 2\pi \left( \frac{x^2 a^2}{2} - \frac{x^4}{4} - 2d \frac{x^3}{3} - d^2 x^2 \right) \]

(shifting fn.)

\[ \text{(HW)} \quad \frac{(a-d)^3(3a+d)}{12} \]

(can also use shells to rotate about x-axis if the transformed function)

5.4. Length of a curve

Find length \( L \) of curve of \( f(x) \) for \( a \leq x \leq b \)

\[ \Delta l \approx \Delta x \sqrt{\left( \frac{dx}{dx} \right)^2 + \left( \frac{dy}{dx} \right)^2} \]

\( \Delta l \approx \frac{dx}{dx} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \)

\[ y = f(x) \quad \Rightarrow \quad \frac{dy}{dx} = f'(x) \quad \Rightarrow \quad \Delta l = dx \sqrt{1 + (f'(x))^2} \]

\[ L = \int_{a}^{b} \sqrt{1 + (f'(x))^2} \, dx \quad ( = \int_{0}^{a} \rho \, ds ) \]
Ex. Consider \( y = f(x) = x^3 + \frac{1}{12x} \). Find length \( L \) for \( 1 \leq x \leq 2 \).

\[
\begin{align*}
\frac{dy}{dx} &= 3x^2 - \frac{1}{12x^2} \\
(f'(x))^2 &= 9x^4 - \frac{1}{2} + \frac{1}{144x^4} \\
L &= \int_1^2 \sqrt{1 + 9x^4 - \frac{1}{2} + \frac{1}{144x^4}} \, dx \\
&= \int_1^2 \sqrt{(3x^2 + \frac{1}{12x^2})^2} \, dx = (x^3 - \frac{1}{12x}) \bigg|_1^2 = 8 - \frac{1}{24} - 1 + \frac{1}{12} = \frac{7}{24}
\end{align*}
\]

Ex. Consider \( y = f(x) = x^2 \). Find \( L \) for \( 1 \leq x \leq 2 \).

\[
\frac{dy}{dx} = 2x \Rightarrow (f'(x))^2 = 4x^2
\]

\[
L = \int_1^2 \sqrt{1 + 4x^2} \, dx = \ldots \quad \text{need all tools from ch. 6}
\]

solution: \( L = \left( \frac{1}{2} x \sqrt{1 + 4x^2} + \frac{1}{4} \ln \left| \sqrt{1 + 4x^2} + 2x \right| \right) \bigg|_1^2 = \ldots \)
Review: difference between centre of mass, $\bar{x}$, and point where to cut a piece into two pieces of equal weight, $x_{1/2}$.

$$\bar{x} = \frac{b+a}{2}$$

$\bar{x}_{1/2} = \bar{x}$ (distribution symmetric about $\bar{x}$)

$\bar{x}_{1/2}$ unchanged

$$\bar{x} > x_{1/2}: \text{centre of mass is shifted towards the 'tail'}$$

$\Rightarrow$ dikrit (use $a=0 \Rightarrow x_{1/2} = \frac{b}{2}, \bar{x} = \frac{5}{8}b$)
chapter 6: Techniques of integration

→ collection of tricks to find anti-derivatives
→ anti-derivatives are defined only up to a constant

\[ \int f(x) \, dx = F(x) + C \text{, with } \frac{df}{dx} = f(x) \]

 indefinite integral

6.1 Differential Notation

→ for \( f(x) = y \) how does a small change in \( x \) translate into small changes in \( y \)?

\[ \frac{dy}{dx} = f'(x) \]

\[ \int dy = \int f'(x) \, dx \]

\[ y = f(x) + C \]

6.2 Substitution

→ reverse of chain rule for derivatives

recall: \( \frac{d}{dx} F(u(x)) = \frac{dF(u)}{du} \cdot \frac{du}{dx} = F'(u(x)) \cdot u'(x) \)

\[ = f(u(x)) \cdot u'(x) \text{ with } \frac{dF(x)}{dx} = f(x) \]
reverse \[ \int f(u(x)) \cdot u'(x) \, dx = \int f(u(x)) \cdot \frac{du(x)}{dx} \, dx \]

\[ = \int f(u) \, du = \int \frac{dF}{du} \, du = \int dF = F(u(x)) + C \]

\[ \Rightarrow \text{substitution: if integrand is of form } \text{"function of another function times derivative of latter function"} \]

**Ex. 1** \( \int x \cdot \cos(x^2) \, dx = ? \quad u = x^2 \implies \frac{du}{dx} = 2x \quad \rightarrow \quad dx = \frac{du}{2x} \)

\[ = \int x \cdot \cos(u) \frac{1}{2x} \, du = \frac{1}{2} \sin(u) + C = \frac{1}{2} \sin(x^2) + C \]

**Ex. 2** \( \int \sin^3 x \cdot \cos x \, dx = ? \quad u = \sin x, \quad \frac{du}{dx} = \cos x \)

\[ = \int u^3 \cdot \cos x \frac{1}{\cos x} \, du \quad dx = du \cdot \frac{1}{\cos x} \]

\[ = \frac{u^4}{4} + C = \frac{1}{4} \sin^4 x + C \]

**Ex. 3** \( \int \frac{1}{x \cdot \ln x} \, dx = ? \quad u = \ln x \quad \frac{du}{dx} = \frac{1}{x} \quad dx = x \cdot du \)

\[ = \int \frac{1}{x} \cdot u \, du = \ln |u| + C = \ln |\ln x| + C \]

**Ex. 4** \( \int \frac{x^2}{\sqrt{1-x^3}} \, dx = ? \quad u = 1-x^3 \quad \frac{du}{dx} = -3x^2 \quad dx = -\frac{1}{3x^2} \, du \)

\[ = -\frac{1}{3} \int \frac{x^2}{u^{1/2}} \frac{1}{x^2} \, du = \]
\[-\frac{2}{3} \sqrt{u} + C = -\frac{2}{3} \sqrt{1 - x^3} + C\]

a) use orig. var.: \(\left(-\frac{2}{3} \sqrt{1 - x^3} + C\right)|_0^1 = -\frac{2}{3} + \frac{2}{3} \sqrt{2}\)

b) change boundaries: \(\left(-\frac{2}{3} u' + C\right)|_2^1 = -\frac{2}{3} + \frac{2}{3} \sqrt{2}\)

Ex. 5. \(\int \frac{x^5}{1-x^3} \, dx = ? \quad u = 1-x^3, \quad \frac{du}{dx} = -3x^2\)

\[dx = \frac{-1}{3x^2} \, du\]

\[-\frac{1}{3} \int \frac{x^3}{1-u} \cdot \frac{1}{x^2} \, du\]

\[\text{problem! both } u \text{ and } x\]

\(\Rightarrow \text{use } x = \sqrt[3]{1-u} = (1-u)^{\frac{1}{3}} \rightarrow x^3 = 1-u\)

\[-\frac{1}{3} \int \frac{1-u}{1-u} \, du = \frac{1}{3} \int u - \frac{1}{1-u} \, du = \left(\frac{1}{3}\right)

\[= -\frac{2}{3} \sqrt{1-x^3} (2 + x^3) + C\]

Summary: Substitution

indefinite integral: \(\int f(g(x)) \cdot g'(x) \, dx\)

1) find substitution: \(u = g(x)\)

2) find new differential: \(\frac{du}{dx} = g'(x) \rightarrow dx = \frac{1}{g'(x)} \, du\)

3) rewrite integral: \(\int f(u) \, du\)

\(\rightarrow \text{new integral should be simpler}\)

\(\rightarrow \text{original variable must be eliminated}\)

\(\rightarrow \text{no guarantee of success}\)

4) solve new integral: \(F(u) + C \quad \text{don't forget!}\)
5) rewrite in terms of orig. var: \( F(g(x)) + C \)

definite integrals: \( \int_{a}^{b} f(g(x)) g'(x) \, dx \)

\( \rightarrow \) two variants:

a) 'change bounds': steps 1-4 above
   - first new bounds: \( a \rightarrow g(a) \), \( b \rightarrow g(b) \)
   - evaluate at new bounds: \( \frac{F(b)}{g(b)} - \frac{F(a)}{g(a)} \)

\( \rightarrow \) never flip bounds