\[
\lim_{x \to 0^+} \frac{2 \sin x \cdot \cos x}{3x^2} = \frac{2 \cos^6 x}{3} \lim_{x \to 0^+} \frac{\sin x}{x^2} \quad \text{(L'Hôpital rule)}
\]

\[
= 0 \text{ (useless, tricky)}
\]

\[
\lim_{x \to 0^+} \frac{\sin x}{x^2} = \lim_{x \to 0^+} \frac{\cos x}{2x} 
\]

\[
\Rightarrow \text{limit does not exist; diverges to } +\infty
\]

Note: Initially applying L'Hôpital again to *

\[
\lim_{x \to 0^+} \frac{\cos x}{2x} \times \lim_{x \to 0^+} \frac{\sin x}{2} = 0
\]

\[
\Rightarrow \text{wrongly suggest that limit exists!}
\]

- for \( x \to 0^- \) then limit diverges to \(-\infty\)
- Had limit existed, the 'harmless' terms still affect the value of limit but not its existence.

Ex. 3) \( \lim_{E \to 0^+} E \cdot \ln E = ? \)

\[
\lim_{E \to 0^+} E \cdot \ln E = \lim_{E \to 0^+} \frac{E \ln E}{1/\epsilon}
\]

\[
\Rightarrow \text{L'Hôpital rule}
\]

\[
= \lim_{E \to 0^+} \frac{E}{1/E} = \lim_{E \to 0^+} \frac{E^2}{\epsilon} = 0
\]

\[
\Rightarrow \int_0^\epsilon \ln(x^2) \, dx = 2e^2
\]
Chapter 8: Continuous probability distributions

8.1 Introduction:
- probabilistic processes
  - coin toss, dice,
  - eye color or gender
- discrete probability
- weight of chocolate bar
- time until decay of radioactive atom
  → continuous probabilities (cont. range of outcomes)
- events occur with prob.
  - 'heads' happens with prob. 0.5
    → discr. prob.
  - prob. that chocolate is between 89 g - 101 g
    or 100 g ± 1 g
  - prob. that radioactive atom has decayed within 1 year

Illustration: consider prob. of failure of light bulbs over time.

Typical questions:
a) prob. that bulb fails within warranty?
\[ P(X < 10) = \int_0^{10} p(t) \, dt \]
\[ \text{event (fails before 10 years) } \quad \text{prob. that bulb fails between } t \text{ and } t + dt \]
\[ \rightarrow p(t) \text{ is probability density function, pdf} \]

b) prob. that bulb fails after exactly 12 years?
\[ P(X = 12) = \int_{12}^{12} p(t) \, dt = 0 \]

c) Find \( \int_0^{\infty} p(t) \, dt = 1 \)
\[ \rightarrow \text{prob. that bulb will eventually fail} \]

d) prob. that bulb lasts longer than 15 years?
\[ P(X > 15) = \int_{15}^{\infty} p(t) \, dt = \int_{15}^{\infty} p(t) \, dt - \int_0^{15} p(t) \, dt \]

e) avg. lifespan of bulb? \( E \approx 14 \text{ years (from graph)} \)
\[ E = \frac{\int_0^{\infty} t \cdot p(t) \, dt}{\int_0^{\infty} p(t) \, dt} = \frac{\int_0^{\infty} t \cdot p(t) \, dt}{1} \]

Note: Properties of pdf's

- \( p(x) \) is pdf for events \( a \leq x \leq b \) - if
  1) \( p(x) > 0 \) for \( a \leq x \leq b \) (outside of this doesn't matter)
  2) \( \int p(x) \, dx = 1 \) (normalization)
\( p(x) > 1 \) is perfectly fine b/c \( p(x) \) is probability.

\[ P(X < t) = \int_0^t p(s) \, ds = F(t) \leftarrow \text{defines new function} \]

\[ \frac{dF}{dt} = p(t) \quad \text{(FTC)} \]

\( F(t) \) : cumulative distribution function, cdf

\[ \rightarrow \text{cdf is increasing function.} \]

6) Prob. that bulb fails between 10 and 15 years?

\[ P(10 < X < 15) = \int_{10}^{15} p(t) \, dt = \int_0^{15} p(t) \, dt - \int_0^{10} p(t) \, dt \]

\[ = F(15) - F(10) \quad \text{(with} \quad \frac{dF}{dt} = p(t) \text{)} \]

Properties of cdf:

- \( F(x) \) is cdf for events between \( a \) and \( b \)
  - if \( \frac{dF}{dx} = p(x) \) is pdf for \( a \leq x \leq b \)

\[ \rightarrow F(a) = 0, \quad F(b) = 1 \]

8.2 Mean, Moments & Median

Consider \( p(x) \) as pdf for \( a \leq x \leq b \)

\[ \int_a^b p(x) \, dx = 1 \quad \rightarrow \text{normalization (total mass)} \]
2) \( \int_{a}^{b} x \cdot p(x) \, dx = \bar{x} \rightarrow \text{mean, average (center of mass)} \)

\( \rightarrow \text{generalize} \rightarrow \text{moments} \)

3) moments: \( k \)-th moment \( M_k = \int_{a}^{b} x^k \cdot p(x) \, dx \)

\( \rightarrow \text{discrete distribution} \)

\( k = 0: \text{normalization} \)
\( k = 1: \text{mean} \)
\( k = 2: \text{spread/width} \)
\( k = 3: \text{skew/symmetry} \)
\( k = 4: \text{weight of tails} \)

4) Variance – width of distribution

\( \rightarrow \) quantifies spread around \( \bar{x} \)

\[
\text{Var} = \int_{a}^{b} (x - \bar{x})^2 \cdot p(x) \, dx
\]

\( \text{Var} = \int_{a}^{b} (x^2 - 2\bar{x}x + \bar{x}^2) \cdot p(x) \, dx \)

Note: \( \bar{x} = \int_{a}^{b} x \cdot p(x) \, dx \) is known (calculate independently)

\[
= \int_{a}^{b} x^2 \cdot p(x) \, dx - 2\bar{x} \int_{a}^{b} x \cdot p(x) \, dx + \bar{x}^2 \int_{a}^{b} p(x) \, dx
\]

Second moment \( M_2 \)

First moment \( M_1 = \bar{x} \)

Zeroth moment \( M_0 = 1 \)
\[ \frac{\frac{2}{a} \int x^2 \cdot p(x) \, dx - \bar{x}^2}{\sigma} = M_2 - (\bar{M})^2 \]

5) Standard deviation $\sigma$

\[ \sigma = \sqrt{\text{Var}} \rightarrow \text{same units as event} \]

Ex. prob. that event lies within one $\sigma$ of $\bar{x}$:

\[ \int_{\bar{x} - \sigma}^{\bar{x} + \sigma} p(x) \, dx \]

$\bar{x} - \sigma$

→ Historically, variance is more important

(more useful for calculating $\frac{\bar{x}}{\sigma}$ was tedious)

→ $\sigma$ is much more convenient and intuitive

6) Median

→ recall: where to cut piece into two of equal weight?

→ for pdf's this is the median

\[ \int_{x_{\text{med}}}^{x} p(x) \, dx = \int_{x}^{\infty} p(x) \, dx = \frac{1}{2} \int_{\infty}^{\infty} p(x) \, dx = \frac{1}{2} \]

\[ \int_{x_{\text{med}}}^{x} \]

or \[ F(x_{\text{med}}) = \frac{1}{2} \]

→ mean and median are the same if pdf is symmetrical about $\bar{x}$.

Note: mean vs. median: any wealth is meaningless because distribution is skewed (30-40 richest people own the same as the poorer 50% of humans)
8.3 Uniform probability distribution

\[ p(x) = \begin{cases} \text{const.} & \text{as } x \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases} \]

→ all events between \( a \) and \( b \) are equally likely

Find a) moments \( M_0, M_1, M_2 \)
b) mean, variance, std. dev.
c) median

\[ p(x) = \frac{1}{b-a} \text{ for } a \leq x \leq b \]

\[ \int_a^b p(x) \, dx = 1 \]

\[ = c \cdot x \bigg|_a^b = c(b-a) \]

\[ \Rightarrow c = \frac{1}{b-a} \]

a) moments:

\[ M_0 = \int_a^b p(x) \, dx = 1 \quad (\text{see above}) \]

\[ M_1 = \int_a^b x \cdot p(x) \, dx = \frac{1}{b-a} \frac{x^2}{2} \bigg|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2} \]

\[ M_2 = \int_a^b x^2 \cdot p(x) \, dx = \frac{1}{b-a} \frac{x^3}{3} \bigg|_a^b = \ldots = \frac{b^2 + ab + a^2}{3} \]

\[ \text{Check it} \]

b) mean: \( \bar{x} = M_1 = \frac{b+a}{2} \)

Variance: \( \text{Var} = M_2 - (M_1)^2 = \frac{b^2 + ab + a^2}{3} - \left( \frac{b+a}{2} \right)^2 \)

\[ = \ldots = \frac{(b-a)^2}{12} \]

Standard dev.: \( \sigma = \sqrt{\text{Var}} = \frac{b-a}{\sqrt{12}} \)
\[
\text{median: } X_{\text{med}} \rightarrow \int_{a}^{X_{\text{med}}} p(x) \, dx = \frac{1}{2}
\]
\[
\int_{a}^{X_{\text{med}}} \frac{1}{b-a} \, dx = \frac{1}{b-a} (X_{\text{med}} - a) = \frac{1}{2} \rightarrow X_{\text{med}} = \frac{a+b}{2}
\]

→ as expected (since \( b/2 \) pdf symmetric about \( \overline{x} \))

HW: exponential distribution: \( p(t) = \lambda e^{-\lambda t} \) for \( t \geq 0 \)

→ e.g. radioactive decay
(median called half-life of radioactive substance)

8.4. Functions and pdfs

→ many functions satisfy criteria of pdfs (or can be made to do)

Ex 1) Is \( f(x) = 2 - x^2 \) a pdf?

→ no \( b/2 f(x) \leq 0 \) for some \( x \).

→ restrict function to domain where \( f(x) \geq 0 \)

\[ f(x) \geq 0 \text{ for } -\frac{\sqrt{2}}{2} \leq x \leq \frac{\sqrt{2}}{2} \]

→ maximum range of events

→ is \( f(x) \) normalized over range of events?

\[
\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} f(x) \, dx = \int_{0}^{\frac{\sqrt{2}}{2}} 2 - x^2 \, dx = 2 \left[ x - \frac{1}{3} x^3 \right]_{0}^{\frac{\sqrt{2}}{2}}
\]

\[= 4 \frac{\sqrt{2}}{2} - \frac{2}{3} \frac{\sqrt{2}}{2} = \frac{8}{3} \frac{\sqrt{2}}{2} \neq 1
\]

→ not normalized! → easy to fix.

\[\Rightarrow p(x) = C \cdot f(x) = \frac{3}{8\sqrt{2}} (2 - x^2) \text{ for } -\frac{\sqrt{2}}{2} \leq x \leq \frac{\sqrt{2}}{2}
\]

→ tuned \( f(x) \) into pdf.
Ex. 2) Is \( f(x) = \frac{1}{1+x^2} \) pdf for \( x \geq 0 \)
- \( f(x) \geq 0 \) holds for \( x \geq 0 \) \( \checkmark \)
- \( \int_{0}^{\infty} \frac{1}{1+x^2} \, dx = \arctan x \bigg|_{0}^{\infty} = \lim_{x \to \infty} \arctan x - \arctan 0 = \frac{\pi}{2} \)

\[
\frac{\pi}{2} \quad \rightarrow \text{not normalized.}
\]

\[
x
\]
\[
-\frac{\pi}{2}
\]

\[\Rightarrow p(x) = \frac{2}{\pi} \frac{1}{1+x^2} \] is pdf for \( x \geq 0 \)

8.5 Transforming pdfs
Ex. transform pdf of radii \( r \) of bacteria into pdf of volume of bacteria
- Assume spherical bacteria and uniformly distributed radii between 0 and \( R \)
- Mean and median of bacteria

1) pdf of radii
\[ p(r) \]

\[ \rightarrow p(r) = \frac{1}{R} \]

\[ \Rightarrow \text{is volume uniformly distributed too?} \]

\[ \rightarrow \text{No. } \frac{d}{dr} V \propto r^3 \]

2) relate radius and volume: \( V = \frac{4}{3} \pi r^3 \) \( \Rightarrow r = \left( \frac{3V}{4\pi} \right)^{\frac{1}{3}} \)
3) cdf of radii: \( F(r) = \int_0^r p(s) \, ds = \frac{r}{R} \)

4) express cdf in terms of volume: \( F(V) = \frac{1}{R} \left( \frac{3}{4\pi} \right)^{1/3} V^{1/3} \)

\[ V_{\text{max}} = \frac{4\pi}{3} R^3 \]

5) derive pdf of volume: \( \frac{dF(V)}{dV} = \frac{1}{3R \left( \frac{3}{4\pi} \right)^{1/3}} V^{-2/3} = p(V) \)

\[ \Rightarrow \int p(V) \, dV = 1 \quad \text{(check)} \]

Transformation:

\[ p(r) = \begin{cases} \frac{1}{R} & \text{for } 0 \leq r \leq R \\ \frac{1}{3R \left( \frac{3}{4\pi} \right)^{1/3}} V^{-2/3} & \text{for } 0 \leq V \leq V_{\text{max}} \end{cases} \]

\[ \Rightarrow \text{mean volume: } \bar{V} = \int_0^{V_{\text{max}}} V \cdot p(V) \, dV = \int_0^{V_{\text{max}}} V^{1/3} \, dV = \frac{\pi}{3} R^3 \]
\[ F(\nu) = \frac{1}{2} = \frac{1}{R} \left( \frac{3}{4\pi} \right)^{\frac{1}{3}} V_n^{\frac{1}{3}} \]

\[ \Rightarrow V_n = \frac{\pi}{6} R^3 \quad \text{(check)} \]