Chapter 10
Sequences

10.1 Introduction
This chapter has several important and challenging goals. The first of these is to develop the notion of convergent and divergent sequences. As an application of sequences we present a quick excursion into difference equations and the logistic map which reveals important insights into the ecological dynamics of populations.

10.2 Sequences
In this course, a sequence is an ordered list \((a_0, a_1, a_2, a_3, \ldots)\) of real numbers. By a list we mean something similar to a grocery list \{milk, eggs, bread, bananas, \ldots\}, or the roster of a basketball team, or the digits of a phone number. A list is ordered, if the order, in which the items of the list occurs, is important. Typically a grocery list is not ordered: we see the two grocery lists \{milk, eggs, bread, bananas\} and \{eggs, bananas, milk, bread\} as equal. A phone number is however an instance of an ordered list: the two phone numbers 604-123-4567 and 604-456-7123 are distinct. We will not exactly be speaking about grocery lists or phone numbers in this course mainly because these are finite lists. Together we shall focus on ordered lists (i.e. sequences) which have an infinite number of terms. Arranging the natural numbers according to their natural ordering \(0 < 1 < 2 < 3 < 4 < \ldots\) yields the most important infinite sequence: \((0, 1, 2, 3, 4, \ldots)\). Let us remark that according to these definitions we distinguish the two sequences \((0, 1, 2, 3, \ldots)\) and \((1, 0, 2, 3, \ldots)\) as being distinct.

As a convenient shorthand notation to denote the infinite sequence
\[ (a_0, a_1, a_2, \ldots, a_{2012}, a_{2013}, \ldots) \]
we write \((a_k)_{k \geq 0}\) (or sometimes \((a_k)_{k=0}^{\infty}\)). This notation is especially convenient when the terms of the sequence \(a_0, a_1, a_2, \ldots\) have a predictable form. For instance the sequence of natural numbers \((0, 1, 2, 3, \ldots)\) can be written as \((k)_{k \geq 0}\). Or we could express the sequence \((0, -1, 2, -3, 4, -5, 6, -7, \ldots)\) as \((-1)^k k)_{k \geq 0}\). On the other hand, the repeating sequence \((1, 2, 3, 1, 2, 3, \ldots)\) would, in practice, likely be written ‘as-is’. Nonetheless, we
could concoct a function $f$ with the property that $(f(k))_{k \geq 1} = (1, 2, 3, 1, 2, 3, \ldots)$. This function would, however, not clarify the structure of the sequence further – this structure being already transparent.

10.2.1 The index is a ‘dummy’ variable

It is worth emphasizing that the index $k$ occurring in $(a_k)_{k \geq 0}$ is a ‘dummy variable’ and is totally replaceable by some other symbol or letter. That is, the sequences $(a_k)_{k \geq 0}, (a_\ell)_{\ell \geq 0}, (a_{\spadesuit})_{\spadesuit \geq 0}, (a_{\heartsuit})_{\heartsuit \geq 0}$ are all identical: underneath they are all just a shorthand notation for the same sequence $(a_0, a_1, a_2, \ldots)$.

10.2.2 Putting sequences into ‘closed-form’ $(g(k))_{k \geq 1}$

Functions that map real number onto real numbers, $g : \mathbb{R} \rightarrow \mathbb{R}$, are useful in constructing sequences, e.g. $(g(k))_{k \geq 1}$ is a sequence which has the function $g$ as ‘rule’. We remind the reader that the notation ‘$g : \mathbb{R} \rightarrow \mathbb{R}$’ stands for a function $g$, which takes as input a real number $x \in \mathbb{R}$ and produces the real number $g(x) \in \mathbb{R}$ as output. Let us remark that it can happen that very different functions yield the same sequences. For example, both $h(x) = x$ and $j(x) = x + \sin(\pi x)$ yield the sequence of natural numbers $(k)_{k \geq 0} = (k + \sin(\pi k))_{k \geq 0}$ since $\sin(\pi x)$ vanishes whenever $x$ is an integer (see Figure 10.1). In practice, it sometimes happens that the ‘rule’ generating a sequence is not immediately obvious and it can be hard to express the sequence in the form $(a_k)_{k \geq 0}$. As an example, the sequence

$$(1, 0, 1, \frac{1}{2}, 0, 1, \frac{1}{4}, 0, 1, \frac{1}{8}, 0, 1, \frac{1}{16}, 0, \ldots)$$

has a predictable pattern—however the function defining the ‘rule’ of this sequence will not easily be written down using the usual functions encountered in your previous calculus.
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courses. Hence although we understand the pattern generating the sequence (we could easily write the first thousand terms of the sequence), it would be difficult to exhibit a function defining the sequence’s ‘rule’. Finding the ‘rule’ generating a sequence can be sometimes a very important problem: before one can run computer experiments on a sequence, one must first know how to communicate the sequence to the computer. This will require knowing some deterministic procedure to produce the sequence.

10.2.3 A trick question

Let us take a moment to get into some mischief: if we were feeling somewhat obnoxious, then we could pose the following ‘trick question’: “What are the next three terms $t_4, t_5, t_6$ in the sequence $(t_1, t_2, t_3, \ldots) := (1, 2, 3, \ldots)$?”

Of course, if we weren’t looking for mischief, then we would concede that $t_4 = 4, t_5 = 5, t_6 = 6$, and that the sequence we had in mind was the expected $(1, 2, 3, \ldots) = (k)_{k \geq 1}$. However we could also claim that we really had the sequence $(m(k))_{k \geq 1}$ in mind, where we deviantly defined the polynomial

$$m(x) = \frac{1}{2}(2 - x)(3 - x) - 2(1 - x)(3 - x) + \frac{3}{2}(1 - x)(2 - x).$$

This yields

$$(m(k))_{k \geq 1} = (1, 2, 3, -14, -1, 6, \ldots).$$

Most mathematicians (or calculus professors) are not so devilish nor have such a poor taste in jokes. Hence one can generally assume that the first terms of a sequence are written precisely to aid the reader in discovering the ‘rule’ generating the sequence. So the reader should be at ease – the dots ‘...’ occurring in the sequence $(1, 5, 9, 13, \ldots)$ have a function similar to the literary ‘etc’, meaning ‘and so forth’ or ‘continuing in the same way’, and, in this case, gently direct towards the series $(4k + 1)_{k \geq 0} = (1, 5, 9, 13, 17, 21, \ldots)$.

10.2.4 Examples of Sequences

The following are among the basic examples of infinite sequences:

(i) the constant sequence $(\alpha, \alpha, \alpha, \ldots)$ for any number $\alpha \in \mathbb{R}$.

(ii) the sequence of natural numbers: $(0, 1, 2, \ldots) = (n)_{n \geq 0}$.

(iii) the sequence of ‘harmonic numbers’: $(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots) = \left(\frac{1}{k}\right)_{k \geq 1}$.

(iv) the sequence of ‘geometric numbers’: $(r^0, r^1, r^2, r^3, \ldots) = (r^k)_{k \geq 0}$ for any number $r \in \mathbb{R}$.

(v) an alternating sequence $(+1, -1, +1, -1, \ldots) = ((-1)^k)_{k \geq 0}$.

These next sequences are perhaps a bit more curious:

(vi) the sequence $(1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, \ldots) = (\epsilon_k)_{k \geq 1}$, where $\epsilon_k = 1$ whenever $k$ is a power of 2, and $\epsilon_k = 0$ otherwise.
(vii) the Fibonacci sequence \((1, 1, 2, 3, 5, 8, 13, \ldots) = (f_n)_{n \geq 0}\), where \(f_0, f_1 = 1\) and \(f_{k+2} := f_k + f_{k+1}\) for \(k \geq 0\);

(viii) the sequence of squared integers \((0^2, 1^2, 2^2, 3^2, \ldots) = (0, 1, 4, 9, \ldots) = (k^2)_{k \geq 0}\);

(ix) Newton’s method yields the sequence \((x_k)_{k \geq 0}\) where \(x_{k+1} := x_k - \frac{f(x_k)}{f'(x_k)}\) for a differentiable function \(f : \mathbb{R} \to \mathbb{R}\) and a point \(x_0 \in \mathbb{R}\) (see section 10.5).

(x) Sequence of iterates \((x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots)\), for any function \(f : \mathbb{R} \to \mathbb{R}\) and a point \(x_0 \in \mathbb{R}\) (see Section 10.6).

10.3 Convergent and Divergent Sequences

Convergence and divergence are core concepts in mathematics (pure and applied) and the natural sciences. One might say that a behaviour is convergent if, in the long-run, this behaviour becomes fixed. Divergent behaviour is, in contrast, one which never settles. Our interests in convergence and divergence concern the long-time behaviour of sequences of real numbers. Our emphasis is on concrete examples and determining which basic sequences converge or diverge.

We present two definitions of convergence. The first definition is a precise mathematical statement – it is a piece of real mathematics that took over a hundred years to isolate and refine. However, we shall not require such sharp precision and thus present an alternative, simpler and more intuitive formulation, which will serve as the working definition of course.

Definition 3 (rigorous). The sequence \((a_k)_{k \geq 0}\) converges to the limit \(a_\infty\) as \(k \to \infty\) if, for any \(\delta > 0\), a number \(N(\delta)\) exists such that for all values of \(k > N(\delta)\), the \(|a_k - a_\infty| < \delta\) holds.

The precise definition is provided for the simple reason to illustrate that rigorous mathematical definitions are not menacing. This concept of a convergent sequence is depicted in Figure 10.2. Rephrasing Definition 3 in a more cavalier manner yields our working definition:

Definition 4 (working version). An infinite sequence \((a_k)_{k \geq 0}\) converges to \(a_\infty\) if for every ‘error tolerance’ the terms \(a_k\) of the sequence are eventually indistinguishable, i.e. lie within that given error tolerance, from \(a_\infty\).

In other words, regardless of any preassigned ‘error-tolerance’ the sequence \((a_k)_{k \geq 0}\) is eventually indistinguishable – up to that ‘error-tolerance’ – from the constant sequence \((a_\infty, a_\infty, a_\infty, \ldots)\).

The meaning of the expression ‘error tolerance’ requires clarification. How often do we confuse the numbers 0 and 1 as being equal? One coffee mug rarely looks like no coffee mug. One 99 B-Line bus rarely looks like no 99 B-Line bus. On the other hand, one ant very often looks like no ant, and one coffee mug – as seen from a sufficiently far distance, say from the moon’s surface – will almost always look like no coffee mug. Our eyesight
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\[ a_k \]

\[ a_\infty + \delta \]

\[ a_\infty - \delta \]

\[ a_\infty \]

\[ N(\delta) \]

\[ k \]

Figure 10.2. Plotting the points of a sequence \((a_k)_{k \geq 0}\) which is converging to \(a_\infty\). For the given choice of \(\delta\), the terms \(a_k\) become indistinguishable within the ‘error-tolerance’ \(\delta\) from \(a_\infty\) for all \(k > N(\delta)\).

has limited resolution – on sufficiently small scales we are likely to confuse numbers with each other.

Computers have this difficulty, too. Typically the smallest number they can work with is around \(\epsilon = 2^{-1074}\), unless special precautions are taken by the programmer. This means that \(\epsilon + \epsilon/2\) likely equals \(\epsilon\) (or possibly an equally wrong result of \(2\epsilon\)). Similarly, handheld calculators typically do not have sufficient memory to compute \(174!\), i.e. \(174 \cdot 173 \cdot 172 \cdots 3 \cdot 2 \cdot 1\). Depending on memory capacity and/or measurement accuracy, we are eventually unable to distinguish two numbers which are sufficiently close. More concretely, it is impossible to detect the difference between 3cm and 3.0001cm with a household ruler.

By ‘error tolerance’ we denote a distance \(\delta > 0\) at which two numbers \(x, y\) with \(|x - y| < \delta\) can no longer be distinguished. For instance, one can choose a coarse scale such that two numbers \(x, y\) are indistinguishable if their difference \(|x - y|\) is \(< 1\). In this case, \(\delta = 1\) is our ‘error-tolerance’. Within this tolerance, all numbers 1, 1.5, 1.51, 1.9, 1.99999 are ‘the same’. We would, however, be able to distinguish 1 and 2. Likewise 1 and 20 are distinguishable. With an error tolerance of \(\frac{1}{10}\), the numbers 1 and 1.1 would be distinguishable but 1, 1.01, 1.001, 1.00000001 would all be indistinguishable.

The ‘error-tolerance’ can be arbitrarily small but it is always non-zero. No machine or measurement is perfect or infallible and always has some physical limitation. We already implicitly assumed this when stating that an ‘error-tolerance’ corresponds to a strictly positive number \(\delta > 0\).

With this terminology of ‘error-tolerance’ let us now elaborate on our working definition of convergence. To have a sequence \((a_k)_{k \geq 0}\) converging to the number \(a_\infty\) means: for every ‘error-tolerance’ the terms of the sequence eventually become indistinguishable from \(a_\infty\). If we laid out the terms of the sequence

\[ (a_k)_{k \geq 0} = (a_0, a_1, a_2, \ldots, a_{2013}, \ldots, a_{10000000}, \ldots) \]
and if we saw ourselves ‘walking along’ with the terms of the sequence, i.e. walking
from the first term to the second, to the third, fourth, etc., then there would be a term
in the sequence (a point in time during our walk) after which all future terms would be
indistinguishable within our ‘error-tolerance’ from \( a_\infty \). In Definition 3 this term (or ‘point-in-time’) corresponds to \( N(\delta) \). Note that for a smaller ‘error tolerance’, \( \delta' \) with \( \delta > \delta' > 0 \)
then \( N(\delta) \leq N(\delta') \). In other words, a finer ‘error-tolerance’ will require us to walk further
along the sequence before the terms become indistinguishable.

A basic example of a convergent sequence in which the \( \delta \)’s and \( N(\delta) \)’s can be made
totally explicit is in the harmonic sequence \( \left( \frac{1}{n} \right)_{n \geq 0} \) with \( \lim_{n \to \infty} \frac{1}{n} = 0 \). For any
choice of \( \delta > 0 \) we can set \( N(\delta) = \frac{1}{\delta} \). Thus for \( \delta \) very small, \( N(\delta) = \frac{1}{\delta} \) will be very
large. All this follows from seeing that \( \frac{1}{n} < \delta \) if and only if \( n > \frac{1}{\delta} \).

**Notation**

If a sequence \( (a_k)_{k \geq 0} \) converges to the limit \( a_\infty \), we write

\[
\lim_{k \to \infty} a_k = a_\infty,
\]

or for brevity, if there is no ambiguity, simply

\[
a_k \to a_\infty.
\]

The expression \( k \to \infty \) means “as \( k \) tends to \( \infty \).”

A sequence \( (a_k)_{k \geq 0} \) is divergent if it does not converge to any real number. More
precisely, for every real number \( L \) there exists \( \delta > 0 \) such that infinitely many terms in the
sequence differ from \( L \) by more than \( \delta \). Or, negating the above definitions of convergence,
divergence means that for some ‘error-tolerances’, \( \delta \), some terms in the sequence will al-
ways differ from any \( a_\infty \) by more than \( \delta \), no matter how far we walk along the sequence.

**Terminology**

When describing sequences, the following terminology is useful:

(i) A sequence \( (a_k)_{k \geq 0} \) is **bounded** if there exists a number \( M > 0 \) such that \( |a_k| \leq M \)
for all \( k \geq 0 \).

(ii) A sequence \( (a_k)_{k \geq 0} \) is **monotone** or **monotonic** if either (i) \( a_0 \leq a_1 \leq a_2 \leq \ldots \) or
(ii) \( a_0 \geq a_1 \geq a_2 \geq \ldots \); in case (i) we say the sequence is monotonically **increasing**, and in case (ii) it is monotonically **decreasing**.

For example, the sequence

(i) \( \left( \frac{1}{k} \right)_{k \geq 1} \) is bounded by \( M = 1 \) (actually any \( M \geq 1 \)) and monotonically decreasing;

(ii) \( (\cos(\frac{\pi}{k}))_{k \geq 1} \) is bounded by \( M = 1 \) and monotonically increasing;

(iii) \( \left( \frac{-1}{k} \right)_{k \geq 1} \) is bounded by \( M = 1 \) and is not monotonic;

(iv) of natural numbers \( (1, 2, 3, \ldots) \) is not bounded and is monotonically increasing;
10.3. Convergent and Divergent Sequences

10.3.1 Examples of convergent and divergent sequences

We now present some examples of convergence and divergence to clarify what we have in mind.

(i) For every real number $\alpha$ the constant sequence $(\alpha, \alpha, \ldots)$ converges to $\alpha$. With respect to any ‘error-tolerance’, every term is identical $\alpha$. In this case, for every ‘error-tolerance’ $\delta > 0$, any $N(\delta)$ will suffice.

(ii) The sequence of natural numbers $(0, 1, 2, \ldots) = (k)_{k \geq 0}$ keeps increasing and is hence divergent.

(iii) The sequence of ‘harmonic numbers’ $(1, 1/2, 1/3, 1/4, \ldots)$ converges to $a_\infty = 0$ (see Figure 10.3).

(iv) The sequence of ‘alternating harmonic numbers’ $(1, -1/2, 1/3, -1/4, \ldots) = \left(\frac{(-1)^k}{k}\right)_{k \geq 1}$ also converges to $a_\infty = 0$ (see Figure 10.4).

(v) The alternating sequence $(1, -1, 1, -1, \ldots) = ((-1)^k)_{k \geq 0}$ is divergent because it alternates between the two values 1 and $-1$ (see Figure 10.5).

(vi) For $r \in \mathbb{R}$, the geometric sequence $(r^k)_{k \geq 0}$ has different behaviours depending on

Figure 10.3. The harmonic sequence $\left(\frac{1}{k}\right)_{k \geq 1}$ converges to $a_\infty = 0$ as $k \to \infty$. To illustrate this convergence, the function $\frac{1}{x}$ (dashed red line) is shown, which has $y = 0$ as its asymptote.
Figure 10.4. The alternating harmonic sequence \( \left\{ \frac{(-1)^k}{k} \right\}_{k\geq 1} \) converges to 0 as \( k \to \infty \). The sequence is bounded from above by \( f(x) = \frac{1}{x} \) and from below by \(-f(x)\) (dashed red lines).

\[ a_k = 1 \]
\[ 0.5 \]
\[ 0 \]
\[ -0.5 \]
\[ -1 \]

\[ k \]

Figure 10.5. The alternating sequence \( (\frac{-1}{k})_{k\geq 1} \) diverges. The upper and lower bounds at \( y = 1 \) and \( y = -1 \) (dashed red lines) obviously do not converge to one value and hence the terms of the sequence always remain distinguishable for a sufficiently small ’error-tolerance’ \( \delta > 0 \) (here \( 0 < \delta < 2 \)).

the choice of \( r \):

- \(-1 < r < 1 : (r^k)_{k\geq 0} \) converges to 0 \hspace{1cm} (10.1a)
- \( |r| > 1 : (r^k)_{k\geq 0} \) diverges \hspace{1cm} (10.1b)
- \( r = 1 : (r^k)_{k\geq 0} \) constant sequence \( (1, 1, 1, \ldots) \) \hspace{1cm} (10.1c)
- \( r = -1 : (r^k)_{k\geq 0} = ((-1)^k)_{k\geq 0} \) diverges. \hspace{1cm} (10.1d)
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We beg the reader to not memorize this example—one is much better served to every time forget and then re-derive for oneself.

(vii) A sequence of integers \((a_k)_{k \geq 0}\) \((a_k\) positive or negative i.e. with \(a_k \in \mathbb{Z}\)) is convergent only if the sequence is eventually constant. That is, only if there is some \(N\) such that \(a_k = a_N\) for all \(k \geq N\). To see this, suppose \((a_k)_{k \geq 0}\) converges to \(a_\infty\) and consider looking at the sequence with an ‘error-tolerance’ \(\delta = 1\). With respect to this error tolerance all integers are still distinguishable. So if the sequence’s terms eventually become indistinguishable from \(a_\infty\), then at some point they must all coincide with some integer \(\hat{a}\). But if the sequence eventually becomes constant \((\ldots, \hat{a}, \hat{a}, \hat{a}, \ldots)\), then the sequence converges to \(\hat{a}\), i.e. \(\hat{a} = a_\infty\).

(viii) Consider the sequence \((a_k)_{k \geq 1}\) whose terms are defined to be

\[
    a_k = \begin{cases} 
        2 - \frac{1}{k} & \text{if } k \text{ odd}, \\
        1 + \frac{1}{k} & \text{if } k \text{ even}.
    \end{cases}
\]  

For \(k\) large the sequence \((a_k)_{k \geq 1}\) becomes almost indistinguishable from the sequence \((1, 2, 1, 2, \ldots)\). Since this latter sequence is divergent, we see that our initial sequence \((a_k)_{k \geq 1}\) is divergent.

(ix) The digits of \(\pi\) form a sequence \((3, 1, 4, 1, 5, \ldots)\) which diverges; because \(\pi\) is not a rational number (i.e. there do not exist integers \(p, q\) for which \(\pi = \frac{p}{q}\)) the sequence of digits is not eventually constant. From the previous example, we can conclude that the sequence diverges.

(x) The sequence \((3, 3.1, 3.14, 3.141, 3.1415, \ldots)\) converges to \(\pi\); Indeed the rational numbers \(3, 3.1, \ldots\) are rational approximations to \(\pi\) which get better and better.

The next two examples are designed to test our definitions of convergence and divergence.

(xi) The sequence

\[
    (1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, \ldots)
\]

is divergent. It is perhaps natural to at first believe that this sequence converges to \(0\) because most of the terms are \(0\) and the number of consecutive \(0\)'s is increasing. However for any ‘error-tolerance’ that can distinguish \(0\) and \(1\), \(\delta \leq 1\), the never-ending ‘sporadic’ occurrences of \(1\)'s in the sequence will remain distinguishable from the constant sequence \((0, 0, 0, \ldots)\).

An alternative way to see that the sequence is divergent is to realize that the sequence consists of integer terms. As we’ve seen earlier, a sequence of integers is convergent if and only if the sequence is eventually constant. Evidently the above sequence is not eventually constant.

(xii) The sequence

\[
    (1, 0, \frac{1}{2}, 0, 0, \frac{1}{5}, 0, 0, 0, \frac{1}{9}, 0, 0, 0, 0, \frac{1}{14}, 0, \ldots)
\]  

(10.3)
converges to 0. We know the sequence \((1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)\) converges to 0 – for example because each term is smaller than that of the converging harmonic series (see Figure 10.3). This means that regardless of any prescribed ‘error-tolerance’, the sequence eventually becomes indistinguishable from the constant zero sequence \((0, 0, 0, \ldots)\). Likewise, our sequence \((10.3)\) will eventually become indistinguishable from the constant zero sequence and hence converges.

While the sequence \((10.3)\) has never-ending sporadic occurrences of nonzero positive numbers, the magnitude of these ‘outbursts’ are decaying. Very far into the sequence, say after the ten millionth term, these sporadic outbursts are just ‘whispers’ – they are very nearly zeros themselves.

### 10.3.2 The ‘head’ of a sequence does not affect convergence

For any sequence \((a_k)_{k \geq 0}\) the first, say, million terms can be altered without altering either its convergence or divergence. Moreover, if the original sequence converges to a number \(a_\infty\), then any sequence obtained by altering, again say, the first million terms, yields another sequence but still converging to \(a_\infty\). The convergence or divergence of a sequence is a statement about its terms ‘ad infinitum’, i.e. in the (infinitely) long run.

It is perhaps helpful to consider sequences, \((a_k)_{k \geq 0}\), as having two parts: a head and a tail. The finite sequence \((a_0, \ldots, a_k)\) the is called the head, and the infinite sequence \((a_k, a_{k+1}, a_{k+2}, \ldots)\) the tail. A sequence is like a worm stretching from your feet to the horizon. Every segment of the worm’s torso contains a number. A choice of segment, \(a_k\) (choice of worm-saddle), then determines the head and tail of the worm. The head is the finite part in front of the saddle, and the tail is the infinite part trailing behind the saddle. Convergence concerns only the tail, i.e. what happens to those numbers on the torso as we move further along the worm towards the horizon.

### 10.3.3 Sequences and Horizontal Asymptotes of Functions

Our discussion of convergent and divergent sequences should invoke some memories of your differential calculus course and similar discussions in relation to functions. The very definition of derivative itself requires the convergence of a sequence of secants, i.e. the derivative was originally defined by Newton as the limit

\[
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} =: f'(a).
\]

Alternatively we could use the recipe: tangent = \(\lim\) secants. Here we briefly show how your previous experience with functions and limits can be used in understanding convergent sequences.

If the terms in a sequence \((a_k)_{k \geq 0}\) have the form \(a_k = f(k)\) for some function \(f(x)\) then the sequence converges if and only if the function \(f(x)\) has a horizontal asymptote as \(x \to +\infty\). Specifically, if \(\lim_{x \to \infty} f(x) = L\), then \(a_k \to L\).

As an example, consider the function \(f(x) = \frac{2x^3 - 1}{x^3 + 1}\), which has the line \(y = 2\) as
Figure 10.6. Both the sequence \( \left( \frac{2k^3-1}{k^3+1} \right)_{k \geq 0} \) and the function \( f(x) = \frac{2x^3-1}{x^3+1} \) (dashed red line) converge to the horizontal asymptote \( y = 2 \) (dash-dotted line).

horizontal asymptote, i.e. \( \lim_{x \to \infty} f(x) = 2 \). Consequently the sequence

\[
(f(0), f(1), f(2), \ldots) = (-1, 1/2, 5/3, 53/28, \ldots) = \left( \frac{2k^3-1}{k^3+1} \right)_{k \geq 0}
\]

converges to 2 (see Figure 10.6). Similarly, the function \( g(x) = \frac{x^k}{k!} \) becomes unbounded as \( x \to \infty \). Consequently the sequence \( \left( \frac{k^k}{k!} \right)_{k \geq 0} \) diverges (see Figure 10.7).

Figure 10.7. The function \( g(x) = \frac{x^k}{k!} \) grows without bound (dashed red line) and consequently the sequence \( \left( \frac{k^k}{k!} \right)_{k \geq 0} \) diverges.
10.4 Basic Properties of Convergent Sequences

10.4.1 Convergent sequences are bounded

Suppose \((a_k)_{k \geq 0}\) converges to \(a_\infty\). Then for some ‘error-tolerance’, say \(\delta = 1\), the sequence \((a_k)_{k \geq 0}\) eventually becomes indistinguishable from the constant sequence \((a_\infty, a_\infty, a_\infty, \ldots)\). Suppose further that this occurs at the \(N\)th term of the sequence, \(a_N\). Then all terms occurring after \(a_N\) (i.e. all \(a_k\) with \(k > N\)) are very nearly equal to \(a_\infty\). Specifically, we have \(|a_k| < |a_\infty| + \delta\) for all \(k \geq N\). Therefore, the ‘tail’ of the sequence \((a_k)_{k \geq 0}\) is bounded. The only way that the sequence \((a_k)_{k \geq 0}\) could fail to be bounded is if the ‘head’ of the sequence is unbounded, i.e. the terms \(a_k\) with \(0 \leq k \leq N\). But, of course, this does not happen: the ‘head’ of the sequence consists of only finitely many terms and finitely many finite numbers cannot be unbounded. For example, the following explicit constant \(M\) serves as an upper bound for the terms of \((a_k)_{k \geq 0}\): set \(M = \max\{|a_1|, \ldots, |a_N|, |a_\infty| + 1\}\).

The fact that convergent sequences are bounded is very effective at revealing divergent sequences – all unbounded sequences must diverge. From calculus we know that the divergence of many sequences and ratios can be determined just by knowing that for \(n\) large,

\[
\frac{1}{n^2} \ll \frac{1}{n} \ll 1 \ll \log n \ll \sqrt{n} \ll n \ll n^2 \ll 2^n \ll e^n \ll 10^n \ll n! \ll n^n.
\]

For example, the sequences

(i) \((e^k(1 + k + k^2 + \cdots + k^{2013})^{-1})_{k \geq 0}\),

(ii) \((\frac{k^4 + \log(k + 1)}{\sqrt{5k^7 + 16}})_{k \geq 0}\),

(iii) \((\frac{k^k}{k!})_{k \geq 0}\) (c.f. Figure 10.7),

are all unbounded and therefore divergent. This is easily seen from the fact that in each case the numerator grows much faster than the denominator, i.e. \(1 + k + k^2 + \cdots + k^{2013} \ll e^k\), \(\sqrt{5k^7 + 16} \ll k^4 + \log(k + 1)\), and \(k! \ll k^k\).

10.4.2 How can a sequence diverge?

All convergent sequences are bounded. This means that any unbounded sequence is necessarily divergent. However there also exist plenty of bounded but divergent sequences – being bounded is not sufficient to guarantee that a sequence converges. As a basic example consider the alternating sequence \((1, -1, 1, -1, 1, -1, \ldots)\). Similarly, \((1, -1, 0, 1, -1, 0, 1, -1, 0, \ldots)\) is divergent. In these cases we say the sequence oscillates between different limits. Such persistent oscillations are not possible in monotonic sequences and hence bounded monotonic sequences must converge.
10.4.3 Squeeze Theorem

Theorem 10.1 (Squeeze Theorem). Suppose that $a_k \leq b_k \leq c_k$ for all $k \geq 0$ and that both sequences $(a_k)_{k \geq 0}, (c_k)_{k \geq 0}$ converge to the same limit $L = a_\infty = c_\infty$. Then the sequence $(b_k)_{k \geq 0}$ must converge to the same limit $L$.

You have likely seen applications of the squeeze theorem when discussing limits. For instance, the sequence $(\cos \frac{\pi}{n})_{n \geq 1}$ is ‘squeezed’ between the sequence of ‘harmonic numbers’ $(\frac{1}{n})_{n \geq 1}$ and its ‘reflection’ $(-\frac{1}{n})_{n \geq 1}$. Both harmonic sequences converge to 0. Since

$$-\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}$$

for all $x > 0$. We conclude that $\cos \frac{\pi}{n}$ must converge to 0 as well.

10.5 Newton’s Method: A Reprise

Newton’s method is an ingenious procedure for approximating zeros (or roots) of a given function, i.e. to solve $f(x) = 0$ numerically. This has been discussed in the context of differential calculus in the previous term. Suppose we are given a (differentiable) function $f(x)$ and a point $x_0$ which is close to a root of $f(x)$, i.e. $|f(x_0)|$ is nonzero but small. The tangent line of $f(x)$ at $x_0$ yields a good approximation of $f(x)$ in the close vicinity of $x_0$, namely

$$f(x) \approx L_0(x) = f(x_0) + f'(x_0)(x - x_0).$$

The function $L_0(x)$ – a linear function (or polynomial of degree one) in $x$ – has exactly one zero, say, $x_1$. Equivalently, the graph of $L_0(x)$ intersects the $x$-axis in exactly one point, namely at $x_1$. We can solve for $x_1$ explicitly: $L_0(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) = 0$, and thus $x_1 = x_0 - f(x_0)/f'(x_0)$ (see Figure 10.8). Note that the expression for $x_1$ is

![Figure 10.8](image-url)

**Figure 10.8.** Finding the point $x_1$ at which the linear approximation $L_0(x)$ to $f$ at our initial value $x_0$ intersects the $x$-axis.
defined only if the denominator $f'(x_0)$ is nonzero, i.e. only if our initial value $x_0$ is not a critical point of $f(x)$. In the case that $x_0$ is a critical point, the linear approximation $L_0(x)$ coincides with the constant function $L_0(x) \equiv f(x_0)$ and never intersects the $x$-axis in a single point (if $f(x_0) = 0$, $L_0(x)$ coincides with the $x$-axis and ‘intersects’ everywhere).

Repeating the procedure but starting with $x_1$, yields an approximation $x_2$ with $x_2 = x_1 - f(x_1)/f'(x_1)$. That is, $x_2$ is defined as the intersection with the $x$-axis of the linear approximation of $f(x)$ at $x = x_1$, i.e. a zero of the linear function $L_1(x) = f(x_1) + f'(x_1)(x_2 - x_1)$ (see Figure 10.9). Continuing this process (see Figure 10.10) indefinitely yields a sequence $(x_0, x_1, x_2, \ldots)$ where the terms are given by the recursive relation

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}.$$  \hfill (10.4)

The sequence $(x_0, x_1, x_2, \ldots)$ depends on two factors: the function $f(x)$ and the initial value $x_0$. As discussed above, the sequence is no longer defined if one of the terms $x_k$ is a critical point of $f(x)$.

Given a function $f(x)$, it is difficult to determine \textit{a priori} whether or not an initial value $x_0$ yields a convergent sequence $(x_0, x_1, x_2, \ldots)$. It is an important fact of optimization theory that if all derivatives $f', f'', f''', \ldots$ of the function $f$ are continuous, and if $z$ is a root of $f(x)$, then any initial value $x_0$ taken sufficiently close to $z$ yields a sequence $(x_0, x_1, x_2, \ldots)$ which converges to $z$.

In general, the choice of $x_0$ is very important. Even if $x_0$ is reasonably close to a zero of $f(x)$ the terms of the sequence may be very far from $z$ and even converge to another root altogether. An example of a more complicated but nevertheless converging sequence is shown in Figure 10.11.
10.5. Newton’s Method: A Reprise

Figure 10.10. Iterating the process, Newton’s method yields a sequence \((x_0, x_1, x_2, \ldots)\) converging to a zero of our function \(f\). To better illustrate the convergence of the \(x_n\), only the interval \([x_0, x_1]\) of Figure 10.9 is shown here.

Figure 10.11. Newton’s method for the same function \(f(x)\) as in Figures 10.8-10.10 but a different initial guess \(x_0\). The sequence still converges to one of the zeroes \(z\) of \(f(x)\) but takes somewhat surprising detours before settling down.\(^{58}\)

10.5.1 Examples

First, let us apply Newton’s method to find the one unique zero of the function \(y = x^2\). Since we know the zero already, we are more interested in the sequence that is generated (see Figure 10.12). In particular, note that any \(x_0 \neq 0\) yields a sequence converging to the unique zero \(z = 0\). Also note that the zero is simultaneously a critical point of \(x^2\).

Newton’s method is particularly effective at finding square roots, i.e. the two distinct
zeros of the function $y = x^2 - a$ ($a > 0$). Any initial value $x_0 > 0$ yields a sequence converging to the root $z = +\sqrt{a}$ (see Figure 10.13). For $x_0 > \sqrt{a}$ the sequence is monotonically declining but for $0 < x_0 < \sqrt{a}$ the first iterate, $x_1$, is larger ($x_1 > x_0$) and, more specifically, $x_1 > \sqrt{a}$ holds such that starting with $x_1$ the sequence is again monotonically declining. Similarly, a negative initial value, $x_0 < 0$, yields a sequence converging to the other root $z = -\sqrt{a}$ and the sequence is monotonically increasing if $x_0 < -\sqrt{a}$. 

Figure 10.12. *Newton’s method rapidly converges to the unique zero of $y = x^2$ for any initial point $x_0 \neq 0$."

Figure 10.13. The first few iterates $(x_0, x_1, x_2, \ldots)$ of Newton’s method for $f(x) = x^2 - 1$ converge to the root $x = 1$ for $x_0 = 0.5$. 
Instead of focussing on the roots of \( f(x) \) we can view the above sequence as a way to approximate \( \sqrt{a} \). For instance, let us derive an approximation of \( \sqrt{3} \) or, equivalently, use Newton’s method to find the zeros of the function \( f(x) = x^2 - 3 \). According to Equation (10.4) we get
\[
x_{k+1} = \frac{x_k^2 + 3}{2x_k}.
\]

It is interesting to note that for a rational initial value, Newton’s method yields a sequence of rational numbers converging to the irrational number \( \sqrt{3} \). For example, for \( x_0 = 2 \) the first few terms are \((2, \frac{7}{4}, \frac{97}{56}, \frac{18817}{10864}, \ldots)\). Most notably, the fourth term is already accurate up to seven decimal digits, which is an impressive demonstration of how quickly the rational approximation converges.

### 10.6 Iterated Maps

Newton’s method is an example of an iterated map: Equation (10.4) maps a value \( x_k \) onto another value \( x_{k+1} \) and the procedure is iterated, generating a sequence from a function. In this section we discuss iterated maps from a slightly more general perspective and introduce an interesting and fun technique to graphically generate sequences based on iterated maps, called cobwebs.

To begin, pick your favourite function \( g = g(x) \) and your favourite real number \( v \). Evaluating the function \( g(x) \) at the value \( v \) then yields \( g(v) \) – some other number, which typically is different from our initial \( v \). Now we can feed the function \( g(x) \) this new value \( g(v) \) to obtain \( g(g(v)) \). If we keep doing this the number of brackets involved gets easily out of hands. Because of this we assume in this section the convention of computer scientists and logicians to omit all parenthesis from functions. In other words, we write \( gv \) for \( g(v) \) and \( ggv \) for \( g(g(v)) \) etc. Following this notation, we can now feed the function \( g(x) \) the value \( ggv \) and get \( gggv \). This process of feeding and re-feeding a function \( g = g(x) \) with values \( v, gv, ggv, \ldots \) is called iterating the function \( g(x) \) on the value \( v \). Altogether the process yields the sequence
\[
(v, gv, ggv, gggv, \ldots).
\]

The question of whether such a sequence converges is interesting and often very difficult. In particular, it leads directly into the theory of fractals, Mandelbrot & Julia sets. In Section 10.8 we will get a glimpse of the complexities involved.

A convenient geometric perspective on the sequence of iterates \((v, gv, ggv, gggv, \ldots)\) can be obtained by drawing so-called cobwebs. In order to produce a cobweb, first draw the graph of \( y = g(x) \) as well as the diagonal line \( y = x \). The cobweb is then drawn through an alternating sequence of vertical and horizontal line segments. Our starting point is the point \( v \) on the \( x \)-axis, i.e. the point \((v,0)\). From there we move vertically up until we hit the graph of \( g(x) \). This is the point \((v, gv)\). Now we move horizontally until we reach the diagonal line at the point \((gv, gv)\). These are the first two segments of your cobweb (see Figure 10.14a). Next we move again vertically to the graph \( g(x) \) to find the point \((gv, ggv)\) and then horizontally to meet the diagonal at the point \((ggv, ggv)\) (see Figure 10.14b). This process is iterated (see Figure 10.14c, d): vertical to \((ggv, gggv)\), horizontal to \((ggg, gggv)\), vertical to \((ggg, gggv)\) etc.
Figure 10.14. The technique of cobwebbing. a A horizontal, then vertical translation draws the first leg between \((v, gv)\) and \((gv, ggv)\) in our cobweb. b Another horizontal, then vertical translation gives us \(ggv\). c The next horizontal and vertical translation gives us \(gggv\), etc. d The cobweb eventually approaches a fixed point (which has to lie on the diagonal \(y = x\)).

It can be amusing and illuminating to draw the graphs of some familiar functions and sketch the cobwebs for a few different starting points \(v\). Figure 10.15 shows the cobweb for \(y = \log(1 + x)\), where all iterates converge to the origin. A more complicated cobweb is shown for \(y = \cos \left( \frac{7}{2} x \right) \) (see Figure 10.16). Note that the sequence converges to one of the intersection points with the diagonal \(y = x\).

For continuous functions \(g(x)\) with a convergent sequence of iterates \((v, gv, ggv, \ldots)\) an interesting observation can be made. First, let us develop our notation further and write \(g^{[k]}(v)\) for the \(k\)-th iterate of \(g(x)\). For a converging sequence we can then write \(g^{[k]}(v) \to V\) as \(k \to \infty\). By continuity we have

\[
g(V) = g \left( \lim_{k \to \infty} g^{[k]}(v) \right) = \lim_{k \to \infty} \left( g(g^{[k]}(v)) \right) = \lim_{k \to \infty} g^{[k+1]}(v) = V.
\]

That is, \(g(V) = V\). In other words, the limit of a convergent sequence of iterates of a continuous function is always a fixed point. Formally, the point \(p\) is a fixed point of \(g(x)\) if \(g(p) = p\). Graphically, fixed points correspond to points of intersection between the graph of \(y = g(x)\) and the line \(y = x\) (c.f. Figures 10.15-10.16).

However, not all fixed points are stable, which means that no cobweb will eventually converge to this particular point. For example, \(y = \cos \left( \frac{7}{2} x \right)\) has three intersection points with the diagonal \(y = x\) but only the left-most intersection is stable (see Figure 10.16).
10.6. Iterated Maps

\[ x_{t+1} = f(x_t) = \ln(x_t + 1) \]

**Figure 10.15.** Cobweb of \( y = \ln(x + 1) \) showing the iterates of \( f(x) = \ln(x + 1) \) converge to 0.

**Figure 10.16.** After some detours, the cobweb converges to the leftmost point of intersection \( \tilde{z} \) between \( y = \cos(\frac{7}{2}x) \) and the line \( y = x \) for the initial value \( x_0 = -0.343 \).

No matter how close the initial point \( v \) is chosen to one of the other two fixed points, the cobweb will quickly spiral away and eventually converge to the only stable point. In fact, the stability of a fixed point is determined by the angle at which the graph \( g(x) \) intersects the diagonal line. More specifically, the slope of the tangent line to \( g(x) \) at the fixed point \( p \) and the diagonal determines the stability of \( p \). Because \( g(p) = p \), the tangent line to \( g(x) \) at \( p \) is given by \( L_p(x) = mx + b \) with \( m = g'(p) \) and \( b = p(1 - g'(p)) \). If the starting point \( v \) is close to the fixed point, \( |x - p| = \epsilon \ll 1 \), then we can approximate \( g(x) \) by its tangent \( L_p(x) \). Let us now determine under what conditions the point \( L_p(x) \)
lies closer to \( p \), such that \(|L_p(x) - p| < |x - p|\) holds:

\[
|L_p(x) - p| = |mx + b - p| = |g'(p)x + p(1 - g'(p)) - p| = |g'(p)(x - p)|,
\]

and hence the distance to \( p \) will keep decreasing provided that \(|g'(p)| < 1\) holds and hence the sequence converges to \( p \). In particular, if \( 0 < g'(p) < 1 \) the cobweb approaches \( p \) like a ‘staircase’ but in a spiralling fashion if \(-1 < g'(p) < 0\). Conversely, for \( g'(p) > 1 \) the cobweb moves away from \( p \) as a staircase and by spiralling outwards for \( g'(p) < -1 \). The different cases are illustrated in Figure 10.17.

**Figure 10.17.** The stability of a fixed point \( p = g(p) \) is determined by the slope of the tangent line \( L_p(x) \) to \( g(x) \) at \( p \). Here we only show the tangent line (solid red line) and illustrate how its slope affects the convergence to \( p \) or divergence away from \( p \). These observations hold for any function \( g(x) \) in the close vicinity of a fixed point. **a** For \( g'(p) > 1 \) the fixed point \( p \) is unstable and starting a cobweb close to \( p \) at \( v_1 \) and \( v_2 \) results in a staircase that leads away from \( p \). **b** For \( 0 < g'(p) < 1 \) the fixed point is stable and any starting point results in a staircase converging to \( p \). **c** For \( g'(p) < -1 \) the fixed point is unstable but now the cobweb produces a spiral, which results in an alternating sequence that moves further and further away from \( p \). **d** Finally, for \(-1 < g'(p) < 0 \) the fixed point \( p \) is stable but the sequence approaches \( p \) in an oscillatory manner, which results in a cobweb that spirals inwards.
A function may not have any fixed points, i.e. does not intersect the diagonal line, and even if it does, the fixed point may not be stable. Only functions that have at least one stable fixed point can yield convergent sequences at least for some starting points. The cobweb of a convergent sequence is bounded but not all bounded cobwebs are convergent. Just as diverging bounded sequences must have two or more subsequences that converge to different limits, bounded cobwebs that do not converge must visit two or more points repeatedly – this gives rise to periodic cycles. A topic that will be briefly addressed in the next section in the context of population dynamics.

10.7 Difference Equations

Modelling the dynamics of populations is among the most basic problems in theoretical ecology. Let the number of individuals in a population at time $t$ be $N_t$. If, in one unit of time, every individual produces, on average, $\alpha$ offspring then the population size at time $t + 1$ is given by

$$N_{t+1} = \alpha N_t. \tag{10.5}$$

Note that $N_t$ does not need to be an integer number – it represents the average or expected number of individuals in the population. Also note that in sexual populations a simple trick is to count only females because the production of eggs and/or the gestation period usually present the limiting factors for population growth. If the sex ratio is known for the species under consideration then it is straightforward to determine the entire population size. Equation (10.5) is the simplest difference equation. More specifically, the difference in population size at time $t$ and $t + 1$ is given by

$$N_{t+1} - N_t = (\alpha - 1)N_t.$$

Instead of incrementing the time by one unit, we could, for example, increment the time by $\Delta t$ and get

$$N_{t+\Delta t} - N_t = (\alpha^{\Delta t} - 1)N_t.$$

Note that $\alpha$ is the (net) number of offspring per unit time and so there are $\alpha^{\Delta t}$ offspring in the time interval $\Delta t$. Now we can divide both sides by $\Delta t$ and take the limit $\Delta t \to 0$ to recover the differential equation

$$\frac{d}{dt} N(t) = \ln(\alpha) N(t), \tag{10.6}$$

where we have used de l’Hôpital’s rule to calculate the right hand limit. In this limit, the number of offspring per individual (or per capita, i.e. per ‘head’) per time step, $N_{t+1}/N_t = \alpha$, turns into a per capita rate of reproduction $\ln(\alpha)$. This short transformation illustrates that difference equations and differential equations share common features but as we will see they also exhibit very distinct characteristics.

After this brief excursion, let us return to the difference Equation (10.5). So, if there are $N_{t+1}$ individuals at time $t+1$ then, at time $t+2$ the population size is $N_{t+2} = \alpha N_{t+1} =$
By extension, \(k\) time steps later there are \(N_{t+k} = \alpha^k N_t\) individuals. Alternatively, for an initial population size of \(N_0\), the number of individuals at time \(t\) is given by
\[
N_t = \alpha^t N_0.
\] (10.7)

Equation (10.7) provides an explicit solution to the difference Equation (10.7). This is a rare exception as difference equations tend to be more difficult to solve than differential equations. The corresponding solution to the differential Equation (10.6) is \(N(t) = N_0 \alpha^t\).

It is easy to see that for \(\alpha > 1\) the population size keeps increasing without bound. Conversely, for \(\alpha < 1\) the population size decreases and \(N_t\) converges to 0 for \(t \to \infty\), which means the population goes extinct. Only for \(\alpha = 1\) the population size remains constant. More precisely, any population size remains unchanged. However, small changes in the environment or mutations in the offspring result in marginal changes in \(\alpha\) but no matter how small the change is, the population would quickly explode or be driven to extinction.

### 10.8 Logistic Map

In nature, the growth of real populations is, of course, limited by the amount of resources available – the most important resources being food and space. Shortage of resources results in competition among individuals (as well as between species). Let us denote the strength of competition by \(\gamma\). This leads to an improved model for the population dynamics:

\[
N_{t+1} = \alpha N_t - \gamma N_t^2.
\] (10.8)

The per capita number of offspring in each time step is no longer a constant but given by \(N_{t+1}/N_t = \alpha - \gamma N_t\). Hence the reproductive output decreases linearly as the population increases due to competition for resources (see Figure 10.18). Despite its apparent simplicity, this model results in fascinating population dynamics and is know as the logistic difference equation or, simply, the logistic map.

As we have seen before, the population grows if the per capita number of offspring per unit time is greater than 1, shrinks if it is less than 1 and remains constant if it is exactly 1. Equation (10.8) yields a per capita growth of one for \(N^* = (\alpha - 1)/\gamma\). It is easy to verify that for:

(i) \(N_t < N^*\): the per capita growth is > 1 and hence the population size increases;

(ii) \(N_t > N^*\): the per capita growth is < 1 and the population size declines;

(iii) \(N_t = N^*\): the population size does not change. In every time step, each individual produces (on average) one offspring. This means the environment can sustain a population of size \(N^*\) indefinitely. For this reason \(N^*\) is called the carrying capacity and usually denoted by the symbol \(K = N^*\).

For larger and larger populations, fewer and fewer individuals manage to reproduce due to the increased competition. In particular, competition may become so strong that no one reproduces, \(N_{t+1}/N_t = 0\), and hence the population collapses and goes extinct. This occurs for a population of size \(\alpha - \gamma N_t = 0\) or \(N_t = Q = \alpha/\gamma\) (see Figure 10.18). The extinction size \(Q\) represents a hard upper limit for the population size in our model.
For the logistic map, \( N_{t+1} = \alpha N_t - \gamma N_t^2 \) the per capita number of offspring in each time period, \( N_{t+1}/N_t = \alpha - \gamma N_t \) decreases linearly with the population size due to competition for resources (solid red line). \( \alpha \) denotes the (average) maximum number of offspring and \( \gamma \) indicates the strength of selection. The carrying capacity is given by \( K = (\alpha - 1)/\gamma \). Parameters: \( \alpha = 2.5, \gamma = 0.015 \) or \( K = 100 \) and \( Q = 166.6 \).

More specifically, for \( N_t > Q \) the model yields \( N_{t+1} < 0 \), which is obviously biologically meaningless. Further refinements of our model would be necessary to deal with this situation but even so we are able to gain interesting insights. Nevertheless, it is important to be aware of simplifying assumptions and of (biological) limitations of any model. In fact, this may be more important than designing a more complex model that does not suffer from such shortcomings but does not allow a similarly deep and detailed analysis.

An interesting question is now whether the above observations concerning the growth of the population imply that the population size converges to its carrying capacity \( K = N^* \) if we wait long enough. The answer is rather surprising and surprisingly rich.

In order to simplify our discussion, let us first simplify Equation (10.8) by dividing both sides by \( Q \) and substituting \( x_t = N_t/Q \):

\[
x_{t+1} = \alpha x_t (1 - x_t).
\]

(10.9)

This change of variables conveniently reduced the model to a single parameter, \( \alpha \). In this rescaled model, the carrying capacity, \( x^* \), is determined by a per capita growth of one, \( x_{t+1}/x_t = 1 \), or \( \alpha (1 - x^*) = 1 \), which yields \( x^* = 1 - 1/\alpha \) (see Figure 10.19a).

The sequence of population sizes \( (x_0, x_1, x_2, x_3, \ldots, x_t) \) is just the result of the iterated map (10.9), i.e. by repeatedly applying Equation (10.9), given an initial population size of \( x_0 \). This sequence can be derived using the cobwebbing technique introduced in Section 10.6. In particular, the sequence (or the dynamics of the population) depends only on two parameters: the initial population size \( x_0 \) as well as the maximum (average) per capita number of offspring \( \alpha \). In the following we explore the effects of both parameters but the focus is on \( \alpha \) as this turns out to be the more interesting one.

For small \( \alpha \), or more precisely, for \( \alpha < 1 \) the population size dwindles and eventually goes extinct (see Figure 10.19a). This is not surprising as even the maximum per capita
number of offspring cannot maintain the population. Indeed, drawing a cobweb for any initial population size $x_0 \in (0, 1)$ yields a sequence converging to 0. A quick check shows that the logistic map (10.9) has a unique fixed point at $x_0^* = 0$. The slope of the tangent line at the fixed point is $0 < \alpha < 1$, which confirms that the fixed point has to be stable (c.f. Figure 10.17).

Increasing the per capita number of offspring to $\alpha > 1$ the graph of $x_{t+1} = \alpha x_t (1 - x_t)$ rises above the diagonal $x_{t+1} = x_t$ and find $x_0^* = 0$ or $x^* = 1 - 1/\alpha$. The first fixed point, $x_0^* = 0$, we knew already from above, and it marks the extinction of the population. The second fixed point, not surprisingly, refers to the carrying capacity $K$ ($N^* = K$) and any initial population size converges to $x^*$ (see Figure 10.19b) – or, do they?

In order to answer the question of convergence, we first note that the extinction equilibrium is now unstable ($\alpha > 1$). Second, the stability of the fixed point $x^*$, is determined by the slope of the tangent line at $x^*$: $x_{t+1} = f(x_t) = \alpha x_t (1 - x_t)$ and hence

$$f'(x^*) = 2 - \alpha.$$ 

Consequently, $x^*$ is stable if $1 < \alpha < 3$, or $|f'(x^*)| < 1$. This actually implies that if $\alpha > 3$ the carrying capacity becomes unstable! Hence the population size will no longer converge to $x^*$ – but what happens then? This is a fascinating question but unfortunately it goes beyond the scope of this course. Nevertheless, let us catch a glimpse of the fascinating dynamics that unfolds as we increase $\alpha$ further.

**Figure 10.19.** The logistic map $x_{t+1} = \alpha x_t (1 - x_t)$ (solid red line). **a** For $\alpha < 1$ the population goes extinct, regardless of the initial population size $x_0$. **b** For $\alpha > 1$ (but $\alpha < 3$), all populations converge to the stable fixed point $x^* = 1 - 1/\alpha$, which reflects the environments carrying capacity. Even though the trajectories differ for different $x_0$, they all converge to $x^*$.
Figure 10.20. Increasing $\alpha$ produces the period doubling route to chaos. 

a For $2 < \alpha < 3$, the fixed point $x^*$ in Figure 10.19b is still stable but the trajectory approaches the fixed point in an oscillatory manner. 

b For $\alpha$ just above 3, the fixed point $x^*$ becomes unstable and is replaced by a cycle of period two. 

c Increasing $\alpha$ further, renders the two-cycle unstable and it is replaced by a cycle of period four. Further increases of $\alpha$ result in a cascade that doubles the period of stable cycles. 

d For $\alpha_{\text{critical}} \gtrsim 3.56$ the dynamic becomes chaotic. The first 100 iterations of a sample trajectory are shown for $\alpha = 3.99$.

Quite surprisingly the dynamics of this simple system becomes exceedingly complex and was one of the first systems where researchers investigated the onset of chaos, or, more precisely, of deterministic chaos. A brief illustration of the emerging dynamics in summarized Figure 10.20. For $\alpha > 2$, the fixed point $x^*$ remains stable, but the trajectory approaches $x^*$ in an oscillatory manner (the slope at $x^*$ lies between $-1$ and 0, c.f. Figure 10.17). For $\alpha > 3$, the fixed point becomes unstable and is replaced by a cycle of period two (Figure 10.20b). When increasing $\alpha$ further, the two-cycle become unstable as well and is replaced by a period four cycle (Figure 10.20c). Further increases of $\alpha$ produce larger cycles with period 8, 16, 32, 64 ... for smaller and smaller changes of $\alpha$. This cas-
Figure 10.21. Attractions of the logistic map as a function of $\alpha$. Up to $\alpha = 3$ just a single attractor exist, then period two cycles appear (c.f. Figure 10.20b), which results in two attracting points. Larger $\alpha$ lead to cycles of period four (c.f. Figure 10.20c), eight, 16, 32, etc. in rapid succession. However, the number of attractors does not simply keep increasing – even after the threshold to chaos at $\alpha_{\text{critical}} \approx 3.56$ – there are regions with few attractors. For example, for $\alpha$ slightly larger than 3.8, cycles with period 3 occur, followed by another cascade of period doublings. Finally, in the limit $\alpha \to 4$ attractors are distributed along the entire interval $[0, 1]$.

Restricting $x_t$ to the interval $[0, 1]$ corresponds to restricting our population size $N_t$ to the interval $[0, Q]$ (see Figure 10.18). This places an upper limit on our parameter $\alpha$ because if $\alpha$ is too big, the population can explode in one time step which results in population sizes $x_t > 1$ (or $N_t > Q$) but then the competition becomes so strong that in the next time step we get $x_{t+1} < 0$ (or $N_{t+1} < 0$), which is, of course, biologically not meaningful (see Figure 10.22). To prevent $x_t$ and our cobwebs from leaving the interval $[0, 1]$ requires enforcing an upper bound on $\alpha$. The maximum of the function $f(x) = \alpha x (1 - x)$ is located at $\frac{df}{dx} = \alpha (1 - 2x) = 0$ or $x = 1/2$ with $f(1/2) = \alpha/4$. To maintain $x_t \in [0, 1]$ thus requires a maximum of $\leq 1$ or

$$\alpha \leq 4.$$ 

In summary, the logistic map (10.9) yields interesting population dynamics for $1 < \alpha \leq 4$. 
Figure 10.22. For $\alpha > 4$ the cobwebs quickly “shoot off” the graph, which results in negative values that are biologically not meaningful.

10.9 Optional Material

10.9.1 Adding and multiplying convergent sequences

It is very useful to know that term-wise adding and term-wise multiplying of convergent sequences yields other convergent sequences, which converge to the sum and product of the respective limits. More specifically, suppose that we know two sequences $(a_k)_{k \geq 0}$, $(b_k)_{k \geq 0}$, which converge to $a_\infty, b_\infty$, respectively. Then we can conclude that

(i) $\lim_{k \to \infty} ca_k = ca_\infty$, for every constant $c \in \mathbb{R}$,

(ii) $\lim_{k \to \infty} a_k + b_k = a_\infty + b_\infty$,

(iii) $\lim_{k \to \infty} a_k b_k = a_\infty b_\infty$,

(iv) $\lim_{k \to \infty} \frac{a_k}{b_k} = \frac{a_\infty}{b_\infty}$, assuming every $b_k$ and $b_\infty$ is nonzero.

For example, if $a_k \to a_\infty$ and $b_k \to b_\infty$ in the limit $k \to \infty$, then for sufficiently large $k$ both $a_k$ and $b_k$ are close $a_\infty, b_\infty$. Therefore, their sum $a_k + b_k$ is very nearly equal $a_\infty + b_\infty$. All other cases can be motivated by similar, non-rigorous arguments.

10.9.2 Continuous functions and convergent sequences

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and $(a_k)_{k \geq 0}$ a convergent sequence with limit $a_\infty$. Then

$$\lim_{k \to \infty} f(a_k) = f(a_\infty).$$

As example, consider the sequence

$$\left( \sin \left( \frac{2\pi k}{4 + 4k} \right) \right)_{k \geq 0}.$$
We have \( \lim_{k \to \infty} \frac{2\pi k}{4 + 4k} = \frac{\pi}{2} \). Since \( \sin x \) is a continuous function, we know

\[
\lim_{k \to \infty} \sin \left( \frac{2\pi k}{4 + 4k} \right) = \sin \left( \lim_{k \to \infty} \frac{2\pi k}{4 + 4k} \right) = \sin \left( \frac{\pi}{2} \right) = 1.
\]

Therefore, our sequence is convergent with \( \lim_{k \to \infty} a_k = 1 \).
10.10 Exercises

Exercise 10.1 Determine which of the following sequences converge or diverge

(a) \( \{e^n\} \)  (b) \( \{2^{-n}\} \)  (c) \( \{ne^{-2n}\} \)  (d) \( \frac{2}{n} \)  (e) \( \left\{ \frac{n}{2} \right\} \)  (f) \( \{\ln(n)\} \)

Exercise 10.2 Does the sequence \( \left\{ \log \left( \frac{2 + 3k + k^2}{3 + 2k + k^2} \right) \right\}_{k \geq 0} \) converge or diverge? If it converges, what is its limit?

Exercise 10.3 Does the sequence \( \left\{ \frac{2 \sin(k) \cos(2k) + k^2}{k^2 + 3k + 7} \right\}_{k \leq -1} \) converge or diverge? If it converges, what is its limit?

Exercise 10.4 Which of the following sequences are monotonic? Which are bounded?

(a) \( \left\{ e^{-2k} \right\}_{k \geq 0} \)  (b) \( \left\{ \sin(k) \cos(k) \right\}_{k \geq 0} \)  (c) \( \left\{ k^2 \right\}_{k \geq -3} \)  (d) \( \left\{ \frac{k^2 + 1}{k^2 + 2} \right\}_{k \geq 0} \)

Exercise 10.5 Suppose that you deposit $1000 into a bank account that earns 6% annual interest (compounded once yearly). How long will you have to wait until the balance of your account reaches $2000? What is your balance after \( n \) years?

Exercise 10.6 A sequence \( \{a_k\}_{k \geq 0} \) is called arithmetic if \( a_{k+1} - a_k \) does not depend on \( k \); that is, \( a_{k+1} - a_k \) is the same for each \( k \geq 0 \). If \( \{a_k\}_{k \geq 0} \) is an arithmetic sequence satisfying \( a_5 = 16 \) and \( a_7 = 22 \), what is \( a_{30} \)? Can you find a formula for \( a_k \)?

Exercise 10.7 Imagine that two people sit down to share a pie. In the first division of the pie, a third of the pie is given to the first person and a third of what remains is given to the second. This process then repeats forever; in each division, the first person takes a third of the remaining pie and then the second person takes a third of what’s left after that. In the \( k \)th division, what fraction of the total pie does each person eat?

Exercise 10.8 Determine whether or not the following sequences converge (and the limits if they do converge):

(a) \( \left\{ \frac{\ln n}{\ln(\ln n)} \right\}_{n \geq 0} \)  (b) \( \left\{ \sqrt{n^2 + n + 1 - n} \right\}_{n \geq 0} \)

Exercise 10.9 Consider the sequence \( \{a_k\}_{k \geq 0} \) with \( a_0 = 2 \) that satisfies

\[ a_{k+1} = \frac{1}{2} (a_k + 1) \]

for \( k \geq 0 \). Is this sequence bounded (from above or below)? Is it monotone?
Exercise 10.10  Let $a > 0$. Using Newton’s method, write down a sequence $\{x_k\}_{k \geq 0}$ that converges to $\sqrt[3]{a}$. What are the first few terms in this sequence for $a = 2$?

Exercise 10.11  The Gauss map is defined by $x \mapsto e^{-\alpha x^2} + \beta$ for some $\alpha$ and $\beta$. For $\alpha = \beta = 1$, use Newton’s method to approximate the fixed point of this map, i.e., the value $x^*$ such that $x^* = e^{-\alpha(x^*)^2} + \beta$. How did you choose your initial point, $x_0$?

Exercise 10.12  You are contracted to work for a week, working 8 hours a day for 5 days total. In the first hour of work, you earn $7. In each subsequent hour of work your hourly salary goes up by $0.20. How much do you make in your last hour of work for the week?

Exercise 10.13  For the sequence $a_n = \frac{n!}{2^n}$, determine $\lim_{n \to \infty} \frac{a_n}{a_{n+1}}$.

Exercise 10.14  For fixed $a$ and $d$, consider the sequence $a_n := a + nd$. Using

$$\sum_{n=0}^{N} n = \frac{1}{2} N (N + 1),$$

find a formula for $\sum_{n=0}^{N} a_n$. What does the worker of Problem 10.12 earn for the week?

Exercise 10.15  Suppose that you try out a new technique to increase the amount of time you can spend concentrating on your homework. Before you try this new technique, you can spend only 20 minutes working before getting distracted. After each day of using this method, the amount of time you can focus increases by 2%. After some amount of time, you find that you are finally able to spend one uninterrupted hour on your homework before looking for something else to do. Unfortunately, by this point, you’ve forgotten how long ago you started using this new method. For how many days have you been using it?
10.11 Solutions

Solution to 10.1

(a) divergent  (b) convergent  (c) convergent
(d) convergent  (e) divergent  (f) divergent

Solution to 10.2 The sequence is convergent, and its limit is 0.

Solution to 10.3 The sequence is convergent, and its limit is 1.

Solution to 10.4

(a) monotonically decreasing, bounded  (b) bounded, not monotonic
(c) neither bounded nor monotonic  (d) monotonically decreasing, bounded

Solution to 10.5 \( \approx 11.9 \) years; after \( n \) years, the balance is \( 1000 \left( 1.06 \right)^n \)

Solution to 10.6 The formula is \( a_k = 1 + 3k \), and \( a_{30} = 91 \).

Solution to 10.7 In the \( k \)th division, the first person eats \( \left( \frac{1}{3} \right) \left( \frac{2}{3} \right)^{2k} \) of the pie and the second person eats \( \left( \frac{1}{3} \right) \left( \frac{2}{3} \right)^{2k+1} \) of the pie.

Solution to 10.8

(a) diverges  (b) converges (to \( 1/2 \))

Solution to 10.9 The sequence is monotone decreasing, bounded from above by 2, and from below by 1.

Solution to 10.10 \( \{x_n\}_{n \geq 0} \) satisfies \( x_{n+1} = \frac{2x_n^3 + a}{3x_n^2} \). If \( x_0 = a = 2 \), then the first several terms in this sequence are 2, 3/2, 35/27, 125116/99225, \ldots .

Solution to 10.11 The fixed point is \( x^* \approx 1.2237 \).

Solution to 10.12 Your pay for the \( n \)th hour of work is \( a_n = 7 + 0.20 \left( n - 1 \right) \), so your pay for the last hour of the week is \( a_{40} = 14.80 \).

Solution to 10.13 \( \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = 0 \).

Solution to 10.14 \( \sum_{n=0}^{N} a_n = (N + 1) \left( \frac{2a + N d}{2} \right) = (N + 1) \left( \frac{a + a_N}{2} \right) \). The worker earns $436 for the week.
Solution to 10.15  At the end of the 56th day of using this method, you can finally concentrate for one hour.