Chapter 4

Applications of the definite integral to velocities and rates

4.1 Introduction

In this chapter, we encounter a number of applications of the definite integral to practical problems. We will discuss the connection between acceleration, velocity and displacement of a moving object, a topic we visited in an earlier, Differential Calculus Course. Here we will show that the notion of antiderivatives and integrals allows us to deduce details of the motion of an object from underlying Laws of Motion. We will consider both uniform and accelerated motion, and recall how air resistance can be described, and what effect it induces.

An important connection is made in this chapter between a rate of change (e.g. rate of growth) and the total change (i.e. the net change resulting from all the accumulation and loss over a time span). We show that such examples also involve the concept of integration, which, fundamentally, is a cumulative summation of infinitesimal changes. This allows us to extend the utility of the mathematical tools to a variety of novel situations. We will see examples of this type in Sections 4.3 and 4.4.

Several other important ideas are introduced in this chapter. We encounter for the first time the idea of spatial density, and see that integration can also be used to "add up" the total amount of material distributed over space. In Section 5.2.2, this idea is applied to the density of cars along a highway. We also consider mass distributions and the notion of a center of mass.

Finally, we also show that the definite integral is useful for determining the average value of a function, as discussed in Section 4.6. In all these examples, the important step is to properly set up the definite integral that corresponds to the desired net change. Computations at this stage are relatively straightforward to emphasize the process of setting up the appropriate integrals and understanding what they represent.

4.2 Displacement, velocity and acceleration

Recall from our study of derivatives that for $x(t)$ the position of some particle at time t, $v(t)$ its velocity, and $a(t)$ the acceleration, the following relationships hold:

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = a.$$

(**Velocity** is the derivative of position and **acceleration** is the derivative of velocity.) This means that position is an anti-derivative of velocity and velocity is an anti-derivative of acceleration.

Since position, $x(t)$, is an anti-derivative of velocity, $v(t)$, by the Fundamental Theorem of Calculus, it follows that over the time interval $T_1 \le t \le T_2$,

$$\int_{T_1}^{T_2} v(t)\, dt = x(t)\Big|_{T_1}^{T_2} = x(T_2) - x(T_1). \tag{4.1}$$

The quantity on the right hand side of Eqn. (4.1) is a **displacement**,, i.e., the difference between the position at time T_1 and the position at time T_2. In the case that $T_1 = 0, T_2 = T$, we have

$$\int_0^T v(t)\, dt = x(T) - x(0),$$

as the displacement over the time interval $0 \le t \le T$.

Similarly, since velocity is an anti-derivative of acceleration, the Fundamental Theorem of Calculus says that

$$\int_{T_1}^{T_2} a(t)\, dt = v(t)\Big|_{T_1}^{T_2} = v(T_2) - v(T_1). \tag{4.2}$$

as above, we also have that

$$\int_0^T a(t)\, dt = v(t)\Big|_0^T = v(T) - v(0)$$

is the net change in velocity between time 0 and time T, (though this quantity does not have a special name).

4.2.1 Geometric interpretations

Suppose we are given a graph of the velocity $v(t)$, as shown on the left of Figure 4.1. Then by the definition of the definite integral, we can interpret $\int_{T_1}^{T_2} v(t)\, dt$ as the "area" associated with the curve (counting positive and negative contributions) between the endpoints T_1 and T_2. Then according to the above observations, this area represents the displacement of the particle between the two times T_1 and T_2.

Similarly, by previous remarks, the area under the curve $a(t)$ is a geometric quantity that represents the net change in the velocity, as shown on the right of Figure 4.1.

Next, we consider two examples where either the acceleration or the velocity is constant. We use the results above to compute the displacements in each case.

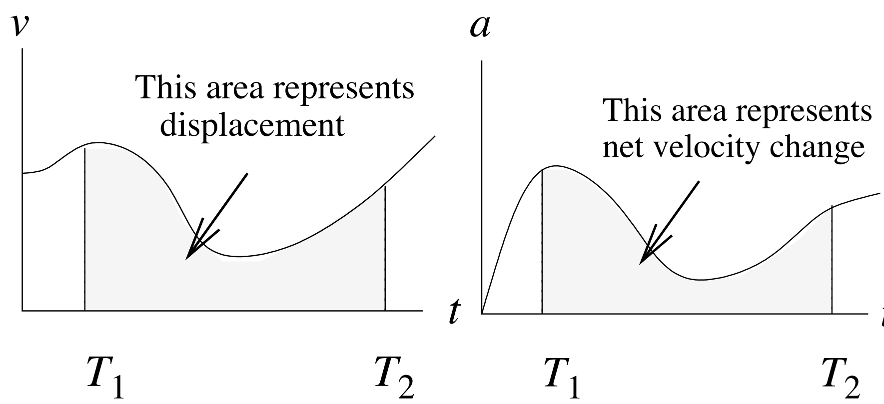

Figure 4.1. *The total area under the velocity graph represents net displacement, and the total area under the graph of acceleration represents the net change in velocity over the interval* $T_1 \le t \le T_2$.

4.2.2 Displacement for uniform motion

We first examine the simplest case that the velocity is constant, i.e. $v(t) = v =$ constant. Then clearly, the acceleration is zero since $a = dv/dt = 0$ when v is constant. Thus, by direct antidifferentiation,

$$\int_0^T v\, dt = vt\Big|_0^T = v(T-0) = vT.$$

However, applying result (4.1) over the time interval $0 \le t \le T$ also leads to

$$\int_0^T v\, dt = x(T) - x(0).$$

Therefore, it must be true that the two expressions obtained above must be equal, i.e.

$$x(T) - x(0) = vT.$$

Thus, for uniform motion, the displacement is proportional to the velocity and to the time elapsed. The final position is

$$x(T) = x(0) + vT.$$

This is true for all time T, so we can rewrite the results in terms of the more familiar (lower case) notation for time, t, i.e.

$$x(t) = x(0) + vt. \tag{4.3}$$

4.2.3 Uniformly accelerated motion

In this case, the acceleration a is a constant. Thus, by direct antidifferentiation,

$$\int_0^T a\, dt = at\Big|_0^T = a(T-0) = aT.$$

However, using Equation (4.2) for $0 \le t \le T$ leads to

$$\int_0^T a\ dt = v(T) - v(0).$$

Since these two results must match, $v(T) - v(0) = aT$ so that

$$v(T) = v(0) + aT.$$

Let us refer to the initial velocity $V(0)$ as v_0. The above connection between velocity and acceleration holds for any final time T, i.e., it is true for all t that:

$$v(t) = v_0 + at. \tag{4.4}$$

This just means that velocity at time t is the initial velocity incremented by an increase (over the given time interval) due to the acceleration. From this we can find the displacement and position of the particle as follows: Let us call the initial position $x(0) = x_0$. Then

$$\int_0^T v(t)\ dt = x(T) - x_0. \tag{4.5}$$

But

$$I = \int_0^T v(t)\ dt = \int_0^T (v_0 + at)\ dt = \left(v_0 t + a\frac{t^2}{2}\right)\Big|_0^T = \left(v_0 T + a\frac{T^2}{2}\right). \tag{4.6}$$

So, setting Equations (4.5) and (4.6) equal means that

$$x(T) - x_0 = v_0 T + a\frac{T^2}{2}.$$

But this is true for *all* final times, T, i.e. this holds for any time t so that

$$x(t) = x_0 + v_0 t + a\frac{t^2}{2}. \tag{4.7}$$

This expression represents the position of a particle at time t given that it experienced a constant acceleration. The initial velocity v_0, initial position x_0 and acceleration a allowed us to predict the position of the object $x(t)$ at any later time t. That is the meaning of Eqn. (4.7)[16].

4.2.4 Non-constant acceleration and terminal velocity

In general, the acceleration of a falling body is not actually uniform, because frictional forces impede that motion. A better approximation to the rate of change of velocity is given by the **differential equation**

$$\frac{dv}{dt} = g - kv. \tag{4.8}$$

[16] Of course, Eqn. (4.7) only holds so long as the object is accelerating. Once the a falling object hits the ground, for example, this equation no longer holds.

We will assume that initially the velocity is zero, i.e. $v(0) = 0$.

This equation is a mathematical statement that relates changes in velocity $v(t)$ to the constant acceleration due to gravity, g, and drag forces due to friction with the atmosphere. A good approximation for such drag forces is the term kv, proportional to the velocity, with k, a positive constant, representing a frictional coefficient. Because $v(t)$ appears both in the derivative and in the expression kv, we cannot apply the methods developed in the previous section directly. That is, we do not have an expression that depends on time whose antiderivative we would calculate. The derivative of $v(t)$ (on the left) is connected to the unknown $v(t)$ on the right.

Finding the velocity and then the displacement for this type of motion requires special techniques. In Chapter 9, we will develop a systematic approach, called Separation of Variables to find analytic solutions to equations such as (4.8).

Here, we use a special procedure that allows us to determine the velocity in this case. We first recall the following result from first term calculus material:

The differential equation and initial condition

$$\frac{dy}{dt} = -ky, \quad y(0) = y_0 \tag{4.9}$$

has a solution

$$y(t) = y_0 e^{-kt}. \tag{4.10}$$

Equation (4.8) implies that

$$a(t) = g - kv(t),$$

where $a(t)$ is the acceleration at time t. Taking a derivative of both sides of this equation leads to

$$\frac{da}{dt} = -k\frac{dv}{dt} = -ka.$$

We observe that this equation has the same form as equation (4.9) (with a replacing y), which implies (according to 4.10) that $a(t)$ is given by

$$a(t) = C\, e^{-kt} = a_0\, e^{-kt}.$$

Initially, at time $t = 0$, the acceleration is $a(0) = g$ (since $a(t) = g - kv(t)$, and $v(0) = 0$). Therefore,

$$a(t) = g\, e^{-kt}.$$

Since we now have an explicit formula for acceleration vs time, we can apply direct integration as we did in the examples in Sections 4.2.2 and 4.2.3. The result is:

$$\int_0^T a(t)\,dt = \int_0^T g\,e^{-kt}\,dt = g\int_0^T e^{-kt}\,dt = g\left[\frac{e^{-kt}}{-k}\right]\Bigg|_0^T = g\,\frac{(e^{-kT}-1)}{-k} = \frac{g}{k}\left(1-e^{-kT}\right).$$

In the calculation, we have used the fact that the antiderivative of e^{-kt} is e^{-kt}/k. (This can be verified by simple differentiation.)

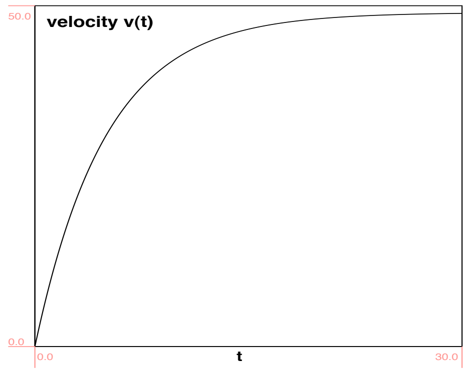

Figure 4.2. *Terminal velocity (m/s) for acceleration due to gravity g=9.8 m/s^2, and $k = 0.2/s$. The velocity reaches a near constant 49 m/s by about 20 s.*

As before, based on equation (4.2) this integral of the acceleration over $0 \le t \le T$ must equal $v(T) - v(0)$. But $v(0) = 0$ by assumption, and the result is true for *any* final time T, so, in particular, setting $T = t$, and combining both results leads to an expression for the velocity at any time:

$$v(t) = \frac{g}{k}\,\left(1 - e^{-kt}\right). \tag{4.11}$$

We will study the differential equation (4.8) again in Section 9.3.2, in the context of a more detailed discussion of differential equations

From our result here, we can also determine how the velocity behaves in the long term: observe that for $t \to \infty$, the exponential term $e^{-kt} \to 0$, so that

$$v(t) \to \frac{g}{k}(1 - \text{ very small quantity}) \approx \frac{g}{k}.$$

Thus, when drag forces are in effect, the falling object does not continue to accelerate indefinitely: it eventually attains a **terminal velocity**. We have seen that this limiting velocity is $v = g/k$. The object continues to fall at this (approximately constant) speed as shown in Figure 4.2. The terminal velocity is also a steady state value of Eqn. (4.8), i.e. a value of the velocity at which no further change occurs.

4.3 From rates of change to total change

In this section, we examine several examples in which the rate of change of some process is specified. We use this information to obtain the total change[17] that occurs over some time period.

[17] We will use the terminology "total change" and "net change" interchangeably in this section.

Changing temperature

We must carefully distinguish between information about the time dependence of some function, from information about the rate of change of some function. Here is an example of these two different cases, and how we would handle them

(a) The temperature of a cup of juice is observed to be

$$T(t) = 25(1 - e^{-0.1t})^\circ\text{Celcius}$$

where t is time in minutes. Find the change in the temperature of the juice between the times $t = 1$ and $t = 5$.

(b) The **rate of change** of temperature of a cup of coffee is observed to be

$$f(t) = 8e^{-0.2t^\circ}\text{Celcius per minute}$$

where t is time in minutes. What is the **total change** in the temperature between $t = 1$ and $t = 5$ minutes ?

Solutions

(a) In this case, we are given the temperature as a function of time. To determine what **net change** occurred between times $t = 1$ and $t = 5$, we find the temperatures at each time point and subtract: That is, the change in temperature between times $t = 1$ and $t = 5$ is simply

$$T(5) - T(1) = 25(1 - e^{-0.5}) - 25(1 - e^{-0.1}) = 25(0.94 - 0.606) = 7.47^\circ\text{Celcius}.$$

(b) Here, we do not know the temperature at any time, but we are given information about **the rate of change**. (Carefully note the subtle difference in the wording.) To get the total change, we would sum up all the small changes, $f(t)\Delta t$ (over N subintervals of duration $\Delta t = (5 - 1)/N = 4/N$) for t starting at 1 and ending at 5 min. We obtain a sum of the form $\sum f(t_k)\Delta t$ where t_k is the k'th time point. Finally, we take a limit as the number of subintervals increases ($N \to \infty$). By now, we recognize that this amounts to a process of integration. Based on this variation of the same concept we can take the usual shortcut of integrating the rate of change, $f(t)$, from $t = 1$ to $t = 5$. To do so, we apply the Fundamental Theorem as before, reducing the amount of computation to finding antiderivatives. We compute:

$$I = \int_1^5 f(t)\,dt = \int_1^5 8e^{-0.2t}dt = -40e^{-0.2t}\Big|_1^5 = -40e^{-1} + 40e^{-0.2},$$

$$I = 40(e^{-0.2} - e^{-1}) = 40(0.8187 - 0.3678) = 18.$$

Only in the second case did we need to use a definite integral to find a net change, since we were given the way that the *rate of change* depended on time. Recognizing the subtleties of the wording in such examples will be an important skill that the reader should gain.

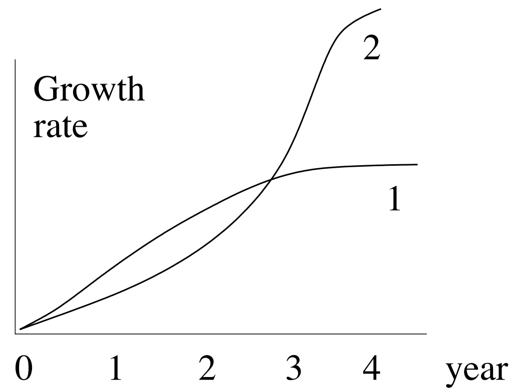

Figure 4.3. *Growth rates of two trees over a four year period. Tree 1 initially has a higher growth rate, but tree 2 catches up and grows faster after year 3.*

4.3.1 Tree growth rates

The rate of growth in height for two species of trees (in feet per year) is shown in Figure 4.3. If the trees start at the same height, which tree is taller after 1 year? After 4 years?

Solution

In this problem we are provided with a sketch, rather than a formula for the growth rate of the trees. Our solution will thus be *qualitative* (i.e. descriptive), rather than *quantitative*. (This means we do not have to calculate anything; rather, we have to make some important observations about the behaviour shown in Fig 4.3.)

We recognize that the net change in height of each tree is of the form

$$H_i(T) - H_i(0) = \int_0^T g_i(t)dt, \quad i = 1, 2.$$

where $i = 1$ for tree 1, $i = 2$ for tree 2, $g_i(t)$ is the growth rate as a function of time (shown for each tree in Figure 4.3) and $H_i(t)$ is the height of tree "i" at time t. But, by the Fundamental Theorem of Calculus, this definite integral corresponds to the area under the curve $g_i(t)$ from $t = 0$ to $t = T$. Thus we must interpret the net change in height for each tree as the area under its growth curve. We see from Figure 4.3 that at $t = 1$ year, the area under the curve for tree 1 is greater, so it has grown more. At $t = 4$ years the area under the second curve is greatest so tree 2 has grown the most by that time.

4.3.2 Radius of a tree trunk

The trunk of a tree, assumed to have the shape of a cylinder, grows incrementally, so that its cross-section consists of "rings". In years of plentiful rain and adequate nutrients, the tree grows faster than in years of drought or poor soil conditions. Suppose the rainfall pattern

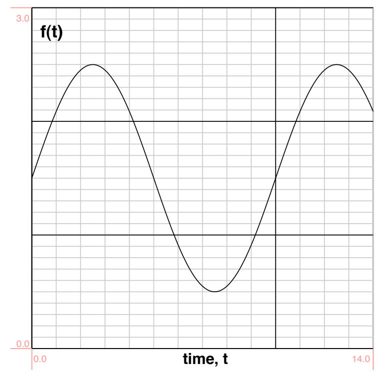

Figure 4.4. *Rate of change of radius, $f(t)$ for a growing tree over a period of 14 years.*

has been cyclic, so that, over a period of 14 years, the growth rate of the radius of the tree trunk (in cm/year) is given by the function

$$f(t) = 1.5 + \sin(\pi t/5),$$

as shown in Figure 4.4. Let the height of the tree trunk be approximately constant over this ten year period, and assume that the density of the trunk is approximately 1 gm/cm^3.

(a) If the radius was initially r_0 at time $t = 0$, what will the radius of the trunk be at time t later?

(b) What is the ratio of the mass of the tree trunk at $t = 10$ years and $t = 0$ years? (i.e. find the ratio mass(10)/mass(0).)

Solution

(a) Let $R(t)$ denote the trunk's radius at time t. The rate of change of the radius of the tree is given by the function $f(t)$, and we are told that at $t = 0$, $R(0) = r_0$. A graph of this growth rate over the first fifteen years is shown in Figure 4.4. The net change in the radius is

$$R(t) - R(0) = \int_0^t f(s)\, ds = \int_0^t (1.5 + \sin(\pi s/5))\, ds.$$

Integrating the above, we get

$$R(t) - R(0) = \left(1.5t - \frac{\cos(\pi s/5)}{\pi/5}\right)\Bigg|_0^t.$$

Here we have used the fact that the antiderivative of $\sin(ax)$ is $-(\cos(ax)/a)$.

Thus, using the initial value, $R(0) = r_0$ (which is a constant), and evaluating the integral, leads to

$$R(t) = r_0 + 1.5t - \frac{5\cos(\pi t/5)}{\pi} + \frac{5}{\pi}.$$

(The constant at the end of the expression stems from the fact that $\cos(0) = 1$.) A graph of the radius of the tree over time (using $r_0 = 1$) is shown in Figure 4.5. This function is equivalent to the area associated with the function shown in Figure 4.4. Notice that Figure 4.5 confirms that the radius keeps growing over the entire period, but that its growth rate (slope of the curve) alternates between higher and lower values.

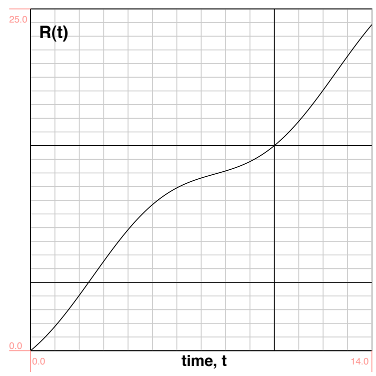

Figure 4.5. *The radius of the tree, $R(t)$, as a function of time, obtained by integrating the rate of change of radius shown in Fig. 4.4.*

After ten years we have

$$R(10) = r_0 + 15 - \frac{5}{\pi}\cos(2\pi) + \frac{5}{\pi}.$$

But $\cos(2\pi) = 1$, so

$$R(10) = r_0 + 15.$$

(b) The mass of the tree is density times volume, and since the density in this example is constant, 1 gm/cm^3, we need only obtain the volume at $t = 10$. Taking the trunk to be cylindrical means that the volume at any given time is

$$V(t) = \pi[R(t)]^2 h.$$

The ratio we want is then

$$\frac{V(10)}{V(0)} = \frac{\pi[R(10)]^2 h}{\pi r_0^2 h} = \frac{[R(10)]^2}{r_0^2} = \left(\frac{r_0 + 15}{r_0}\right)^2.$$

In this problem we used simple anti-differentiation to compute the desired total change. We also related the graph of the radial growth rate in Fig. 4.4 to that of the resulting radius at time t, in Fig. 4.5.

4.3.3 Birth rates and total births

After World War II, the birth rate in western countries increased dramatically. Suppose that the number of babies born (in millions per year) was given by

$$b(t) = 5 + 2t, \quad 0 \le t \le 10,$$

where t is time in years after the end of the war.

(a) How many babies in total were born during this time period (i.e in the first 10 years after the war)?

(b) Find the time T_0 such that the total number of babies born from the end of the war up to the time T_0 was precisely 14 million.

Solution

(a) To find the number of births, we would integrate the birth rate, $b(t)$ over the given time period. The *net change* in the population due to births (neglecting deaths) is

$$P(10)-P(0) = \int_0^{10} b(t)\,dt = \int_0^{10} (5+2t)\,dt = (5t+t^2)|_0^{10} = 50+100 = 150[\text{million babies}].$$

(b) Denote by T the time at which the total number of babies born was 14 million. Then, [in units of million]

$$I = \int_0^T b(t)\,dt = 14 = \int_0^T (5+2t)\,dt = 5T + T^2$$

equating $I = 14$ leads to the quadratic equation, $T^2 + 5T - 14 = 0$, which can be written in the factored form, $(T-2)(T+7) = 0$. This has two solutions, but we reject $T = -7$ since we are looking for time after the War. Thus we find that $T = 2$ years, i.e it took two years for 14 million babies to have been born.

While this problem involves simple integration, we had to solve for a quantity (T) based on information about behaviour of that integral. Many problems in real application involve such slight twists on the ideas of integration.

4.4 Production and removal

The process of integration can be used to convert rates of production and removal into net amounts present at a given time. The example in this section is of this type. We investigate a process in which a substance accumulates as it is being produced, but disappears through some removal process. We would like to determine when the quantity of material increases, and when it decreases.

Circadean rhythm in hormone levels

Consider a hormone whose level in the blood at time t will be denoted by $H(t)$. We will assume that the level of hormone is regulated by two separate processes: one might be the secretion rate of specialized cells that produce the hormone. (The production rate of hormone might depend on the time of day, in some cyclic pattern that repeats every 24 hours or so.) This type of cyclic pattern is called *circadean* rhythm. A competing process might be the removal of hormone (or its deactivation by some enzymes secreted by other cells.) In this example, we will assume that both the production rate, $p(t)$, and the removal rate, $r(t)$, of the hormone are time-dependent, periodic functions with somewhat different phases.

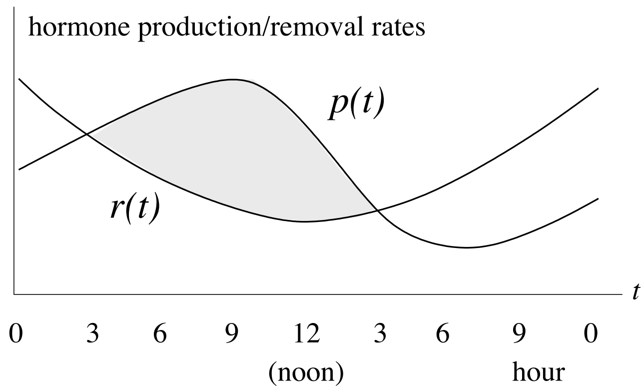

Figure 4.6. *The rate of hormone production $p(t)$ and the rate of removel $r(t)$ are shown here. We want to use these graphs to deduce when the level of hormone is highest and lowest.*

A typical example of two such functions are shown in Figure 4.6. This figure shows the production and removal rates over a period of 24 hours, starting at midnight. Our first task will be to use properties of the graph in Figure 4.6 to answer the following questions:

1. When is the production rate, $p(t)$, maximal?
2. When is the removal rate $r(t)$ minimal?
3. At what time is the hormone level in the blood highest?
4. At what time is the hormone level in the blood lowest?
5. Find the maximal level of hormone in the blood over the period shown, assuming that its basal (lowest) level is $H = 0$.

Solutions

1. We see directly from Fig. 4.6 that production rate is maximal at about 9:00 am.

2. Similarly, removal rate is minimal at noon.

3. To answer this question we note that the total amount of hormone produced over a time period $a \le t \le b$ is
$$P_{\text{total}} = \int_a^b p(t)dt.$$
The total amount removed over time interval $a \le t \le b$ is
$$R_{\text{total}} = \int_a^b r(t)dt.$$
This means that the net change in hormone level over the given time interval (amount produced minus amount removed) is
$$H(b) - H(a) = P_{\text{total}} - R_{\text{total}} = \int_a^b (p(t) - r(t))dt.$$
We interpret this integral as the *area between the curves* $p(t)$ and $r(t)$. But we must use caution here: For any time interval over which $p(t) > r(t)$, this integral will be positive, and the hormone level will have increased. Otherwise, if $r(t) < p(t)$, the integral yields a negative result, so that the hormone level has decreased. This makes simple intuitive sense: If production is greater than removal, the level of the substance is accumulating, whereas in the opposite situation, the substance is decreasing. With these remarks, we find that the hormone level in the blood will be *greatest* at 3:00 pm, when the greatest (positive) area between the two curves has been obtained.

4. Similarly, the least hormone level occurs after a period in which the removal rate has been larger than production for the longest stretch. This occurs at 3:00 am, just as the curves cross each other.

5. Here we will practice integration by actually fitting some cyclic functions to the graphs shown in Figure 4.6. Our first observation is that if the length of the cycle (also called the *period*) is 24 hours, then the *frequency* of the oscillation is $\omega = (2\pi)/24 = \pi/12$. We further observe that the functions shown in the Figure 4.7 have the form
$$p(t) = A(1 + \sin(\omega t)), \quad r(t) = A(1 + \cos(\omega t)).$$
Intersection points occur when
$$\begin{aligned} p(t) &= r(t) \\ A(1 + \sin(\omega t)) &= A(1 + \cos(\omega t)), \\ \sin(\omega t) &= \cos(\omega t)), \\ \Rightarrow \tan(\omega t) &= 1. \end{aligned}$$
This last equality leads to $\omega t = \pi/4, 5\pi/4$. But then, the fact that $\omega = \pi/12$ implies that $t = 3, 15$. Thus, over the time period $3 \le t \le 15$ hrs, the hormone level is

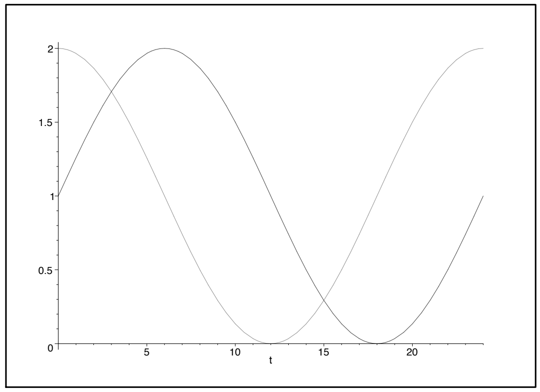

Figure 4.7. *The functions shown above are trigonometric approximations to the rates of hormone production and removal from Figure 4.6*

increasing. For simplicity, we will take the amplitude $A = 1$. (In general, this would just be a multiplicative constant in whatever solution we compute.) Then the net increase in hormone over this period is calculated from the integral

$$H_{\text{total}} = \int_3^{15} [p(t) - r(t)]\,dt = \int_3^{15} [(1 + \sin(\omega t)) - (1 + \cos(\omega t))]\,dt$$

In the problem set, the reader is asked to compute this integral and to show that the amount of hormone that accumulated over the time interval $3 \le t \le 15$, i.e. between 3:00 am and 3:00 pm is $24\sqrt{2}/\pi$.

4.5 Present value of a continuous income stream

Here we discuss the value of an annuity, which is a kind of savings account that guarantees a continuous stream of income. You would like to pay P dollars to purchase an annuity that will pay you an income $f(t)$ every year from now on, for $t > 0$. In some cases, we might want a constant income every year, in which case $f(t)$ would be constant. More generally, we can consider the case that at each future year t, we ask for income $f(t)$ that could vary from year to year. If the bank interest rate is r, how much should you pay now?

Solution

If we invest P dollars (the "principal" i.e., the amount deposited) in the bank with interest r (compounded continually) then the amount $A(t)$ in the account at time t (in years), will grow as follows:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt},$$

where r is the yearly interest rate (e.g. 5%) and n is the number of times per year that interest is compound (e.g. $n = 2$ means interest compounded twice per year, $n = 12$ means monthly compounded interest, etc.) Define

$$h = \frac{r}{n}. \quad \text{Then} \quad n = \frac{r}{h}.$$

Then at time t, we have that

$$\begin{aligned} A(t) &= P(1+h)^{\frac{1}{h}rt} \\ &= P\left[(1+h)^{\frac{1}{h}}\right]^{rt} \\ &\approx Pe^{rt} \quad \text{for large } n \text{ or small } h. \end{aligned}$$

Here we have used the fact that when h is small (i.e. frequent intervals of compounding) the expression in square brackets above can be approximated by e, the base of the natural logarithms. Recall that

$$e = \lim_{h \to 0}\left[(1+h)^{\frac{1}{h}}\right].$$

(This result was obtained in a first semester calculus course by selecting the base of exponentials such that the derivative of e^x is just e^x itself.) Thus, we have found that the amount in the bank at time t will grow as

$$A(t) = Pe^{rt}, \quad \text{(assuming continually compounded interest).} \tag{4.12}$$

Having established the exponential growth of an investment, we return to the question of how to set up an annuity for a continuous stream of income in the future. Rewriting Eqn. (4.12), the principle amount that we should invest in order to have $A(t)$ to spend at time t is

$$P = A(t)e^{-rt}.$$

Suppose we want to have $f(t)$ spending money for each year t. We refer to the *present value* of year t as the quantity

$$P = f(t)e^{-rt}.$$

(i.e. We must pay P now, in the present, to get $f(t)$ in a future year t.) Summing over all the years, we find that the present value of the continuous income stream is

$$P = \sum_{t=1}^{L} f(t)e^{-rt} \cdot \underbrace{1}_{\text{“}\Delta t\text{''}} \approx \int_0^L f(t)e^{-rt}\,dt,$$

where L is the expected number of years left in the lifespan of the individual to whom this annuity will be paid, and where we have approximated a sum of payments by an integral (of a continuous income stream). One problem is that we do not know in advance how long the lifespan L will be. As a crude approximation, we could assume that this income stream continues forever, i.e. that $L \approx \infty$. In such an approximation, we have to compute the integral:

$$P = \int_0^\infty f(t)e^{-rt}\, dt. \tag{4.13}$$

The integral in Eqn. (4.13) is an **improper integral** (i.e. integral over an unbounded domain), as we have already encountered in Section 3.8.5. We shall have more to say about the properties of such integrals, and about their technical definition, existence, and properties in Chapter 10. We refer to the quantity

$$P = \int_0^\infty f(t)e^{-rt}\, dt, \tag{4.14}$$

as the *present value of a continuous income stream* $f(t)$.

Example: Setting up an annuity

Suppose we want an annuity that provides us with an annual payment of $10,000$ from the bank, i.e. in this case $f(t) = \$10,000$ is a function that has a constant value for every year. Then from Eqn (4.14),

$$P = \int_0^\infty 10000e^{-rt}\, dt = 10000\int_0^\infty e^{-rt}\, dt.$$

By a previous calculation in Section 3.8.5, we find that

$$P = 10000 \cdot \frac{1}{r},$$

e.g. if interest rate is 5% (and assumed constant over future years), then

$$P = \frac{10000}{0.05} = \$200,000.$$

Therefore, we need to pay \$200,000 today to get $10,000$ annually for every future year.

4.6 Average value of a function

In this final example, we apply the definite integral to computing the average height of a function over some interval. First, we define what is meant by average value in this context.[18]

Given a function

$$y = f(x)$$

over some interval $a \le x \le b$, we will define average value of the function as follows:

[18]In Chapters 5 and 8, we will encounter a different type of average (also called mean) that will represent an average horizontal position or center of mass. It is important to avoid confusing these distinct notions.

Definition

The average value of $f(x)$ over the interval $a \leq x \leq b$ is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Example 1

Find the average value of the function $y = f(x) = x^2$ over the interval $2 < x < 4$.

Solution

$$\bar{f} = \frac{1}{4-2} \int_2^4 x^2 \, dx = \frac{1}{2} \frac{x^3}{3} \Big|_2^4 = \frac{1}{6} \left(4^3 - 2^3\right) = \frac{28}{3}$$

Example 2: Day length over the year

Suppose we want to know the average length of the day during summer and spring. We will assume that day length follows a simple periodic behaviour, with a cycle length of 1 year (365 days). Let us measure time t in days, with $t = 0$ at the spring equinox, i.e. the date in spring when night and day lengths are equal, each being 12 hrs. We will refer to the number of daylight hours on day t by the function $f(t)$. (We will also call $f(t)$ the day-length on day t.

A simple function that would describe the cyclic changes of day length over the seasons is

$$f(t) = 12 + 4 \sin\left(\frac{2\pi t}{365}\right).$$

This is a periodic function with period 365 days as shown in Figure 4.8. Its maximal value is $16h$ and its minimal value is $8h$. The average day length over spring and summer, i.e. over the first (365/2) $\approx$ 182 days is:

$$\begin{aligned}
\bar{f} &= \frac{1}{182} \int_0^{182} f(t) dt \\
&= \frac{1}{182} \int_0^{182} \left(12 + 4 \sin(\frac{\pi t}{182})\right) \, dt \\
&= \frac{1}{182} \left(12t - \frac{4 \cdot 182}{\pi} \cos(\frac{\pi t}{182})\right) \Big|_0^{182} \\
&= \frac{1}{182} \left(12 \cdot 182 - \frac{4 \cdot 182}{\pi} [\cos(\pi) - \cos(0)]\right) \\
&= 12 + \frac{8}{\pi} \approx 14.546 \text{ hours} \qquad (4.15)
\end{aligned}$$

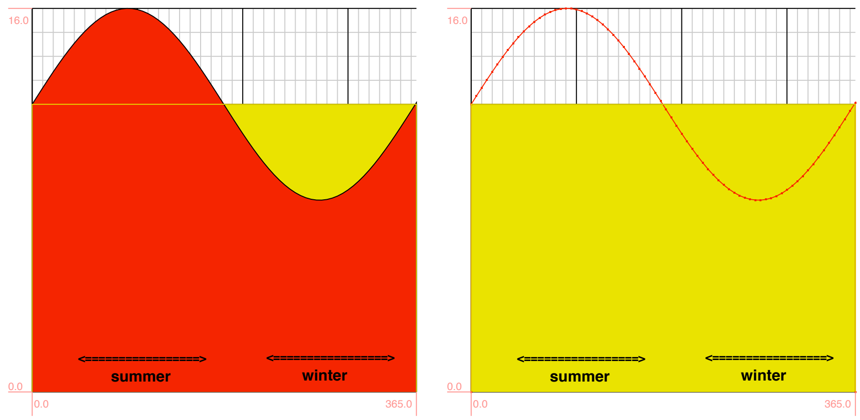

Figure 4.8. *We show the variations in day length (cyclic curve) as well as the average day length (height of rectangle) in this figure.*

Thus, on average, the day is 14.546 hrs long during the spring and summer.

In Figure 4.8, we show the entire day length cycle over one year. It is left as an exercise for the reader to show that the average value of f over the entire year is 12 hrs. (This makes intuitive sense, since overall, the short days in winter will average out with the longer days in summer.)

Figure 4.8 also shows geometrically what the average value of the function represents. Suppose we determine the area associated with the graph of $f(x)$ over the interval of interest. (This area is painted red (dark) in Figure 4.8, where the interval is $0 \le t \le 365$, i.e. the whole year.) Now let us draw in a rectangle over the same interval ($0 \le t \le 365$) having the same total area. (See the big rectangle in Figure 4.8, and note that its area matches with the darker, red region.) The height of the rectangle represents the average value of $f(t)$ over the interval of interest.

4.7 Summary

In this chapter, we arrived at a number of practical applications of the definite integral.

1. In Section 4.2, we found that for motion at constant acceleration a, the displacement of a moving object can be obtained by integrating twice: the definite integral of acceleration is the velocity $v(t)$, and the definite integral of the velocity is the displacement.

$$v(t) = v_0 + \int_0^t a\, ds. \quad x(t) = x(0) + \int_0^t v(s)\, ds.$$

(Here we use the "dummy variable" s inside the integral, but the meaning is, of course, the same as in the previous presentation of the formulae.) We showed that at any time t, the position of an object initially at x_0 with velocity v_0 is

$$x(t) = x_0 + v_0 t + a\frac{t^2}{2}.$$

2. We extended our analysis of a moving object to the case of non-constant acceleration (Section 4.2.4), when air resistance tends to produce a drag force to slow the motion of a falling object. We found that in that case, the acceleration gradually decreases, $a(t) = ge^{-kt}$. (The decaying exponential means that $a \to 0$ as t increases.) Again, using the definite integral, we could compute a velocity,

$$v(t) = \int_0^t a(s)\, ds = \frac{g}{k}(1 - e^{-kt}).$$

3. We illustrated the connection between rates of change (over time) and total change (between on time point and another). In general, we saw that if $r(t)$ represents a rate of change of some process, then

$$\int_a^b r(s)\, ds = \text{Total change over the time interval } a \le t \le b.$$

This idea was discussed in Section 4.3.

4. In the concluding Section 4.6, we discussed the average value of a function $f(x)$ over some interval $a \le x \le b$,

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx.$$

In the next few chapters we encounters a vast assortment of further examples and practical applications of the definite integral, to such topics as mass, volumes, length, etc. In some of these we will be called on to "dissect" a geometric shape into pieces that are not simple rectangles. The essential idea of an integral as a sum of many (infinitesimally) small pieces will, nevertheless be the same.